On MAP Symbol Detection for ISI Channels Using the Ungerboeck Observation Model

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Abstract— In this letter, the well-known problem of a transmission over an additive white Gaussian noise channel affected by known intersymbol interference is considered. We show that the maximum a posteriori (MAP) *symbol* detection strategy, usually implemented by using the Forney observation model, can be equivalently implemented based on the samples at the output of a filter matched to the received pulse, i.e., based on the Ungerboeck observation model. Although interesting from a conceptual viewpoint, the derived algorithm has a practical relevance in turbo equalization schemes for partial response signalling, where the implementation of a whitening filter can be avoided.

Index Terms— Factor graphs (FG), intersymbol interference (ISI), sum-product algorithm (SPA).

I. INTRODUCTION

WHEN the maximum a posteriori (MAP) *sequence* detection strategy is considered for transmissions over known intersymbol interference (ISI) channels, two equivalent approaches for linear modulations can be adopted. The first one is the so-called *Ungerboeck approach* [1]. In this case, the branch metrics of the Viterbi algorithm (VA) implementing this strategy are based on the samples at the output of a filter matched to the received pulse. The second approach is the *Forney approach* [2] and it is based on the output of a whitened matched filter (WMF). Although the relevant VA branch metrics are different, the number of trellis states is the same for both equivalent approaches.

When the MAP *symbol* detection strategy is adopted, as an example to perform turbo equalization [3]–[5], these two observation models do not seem to be equivalent. All the well-known materialization of the MAP *symbol* detection strategy, such as for example the BCJR algorithm [6] (see also [7], [8]) have been obtained by using a probabilistic derivation based on the chain rule and the properties of a Markov source observed through a discrete memoryless channel. Hence, this derivation cannot be directly extended to the case of the Ungerboeck observation model. In this letter, we solve this problem by using a properly defined factor graph (FG) and the sum-product algorithm (SPA) [9].

II. PRELIMINARIES

In this section, we briefly describe the FGs and the SPA, and summarize the results on MAP *sequence* detection for ISI channels.

A. Factor Graphs and the Sum-Product Algorithm

A FG is a bipartite graph which expresses the way a complicated joint probability mass function (pmf) or a joint probability density function (pdf) of many variables factors into the product of *local* functions (not necessarily pmfs or pdfs) [9]. Let $V = \{v_1, \ldots, v_N\}$ denote a set of variables and $F(V)$ a multivariate function. Let $V_1, \ldots, V_m$ denote subsets of V. We say that $F(V)$ admits a factorization with supports $V_1, \ldots, V_m$, if $F(V)$ can be written as the product of the functions $\{F_j : j = 1, \ldots, m\}$, where F_j has the variables in V_j as arguments. The FG representing the factorization $F = \prod_j F_j$ is a *bipartite* graph $\mathcal{G} = \{\mathcal{V}, \mathcal{F}, \mathcal{E}\}$, where nodes in $\mathcal{V}$ (variable nodes) are associated with the variables $v_i \in V$, nodes in $\mathcal{F}$ (factor nodes) are associated with the functions F_j, and there exists an edge $e \in \mathcal{E}$ joining v_i and F_j if and only if $v_i \in V_j$ (i.e., if v_i is an argument of F_j).

Let $F(V)$ be a pmf. Then, if the FG corresponding to the factorization of F has no cycles,[1] the marginal pmfs can be computed *exactly* in a finite number of steps by the SPA [9]. The SPA is defined by the computation rules at variable and at factor nodes, and by a suitable node activation schedule. Denoting by $\mu_{v_i \to F_j}(v_i)$ a message sent from the variable node v_i to the factor node F_j, by $\mu_{F_j \to v_i}(v_i)$ a message in the opposite direction, and by $\mathcal{A}_i$ the set of functions F_j having v_i as argument, the message computations performed at variable and factor nodes are, respectively [9]

$$\mu_{v_i \to F_j}(v_i) = \prod_{H \in \mathcal{A}_i \setminus \{F_j\}} \mu_{H \to v_i}(v_i) \tag{1}$$

$$\mu_{F_j \to v_i}(v_i) = \sum_{\sim \{v_i\}} \left[F_j(\{w \in V_j\}) \prod_{w \in V_j \setminus \{v_i\}} \mu_{w \to F_j}(w) \right] \tag{2}$$

where, following the notation of [9], we indicate by $\sum_{\sim \{v_i\}}$ the *summary* operator, i.e., a sum over all variables excluding v_i. In the absence of cycles in the graph, the computation usually starts at the leaves of the graph and proceeds until two messages have been passed over every edge, one in each direction.

B. MAP Sequence Detection: Ungerboeck and Forney Approaches

The Ungerboeck approach for MAP *sequence* detection [1] is based on the samples $\{x_k\}$ at the output of a filter matched to the received pulse $p(t)$. The sample at time kT, can be

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[1]A *cycle* is a closed path in the graph.

expressed as

$$x_k = \sum_{\ell=-L}^{L} g_\ell a_{k-\ell} + n_k \tag{3}$$

where L is a suitable integer parameter, $\mathbf{a} = \{a_k\}$ is the sequence of M-ary transmitted symbols, assumed independent, it is

$$g_\ell = p(t) \otimes p^*(-t)|_{t=\ell T}$$

and the noise samples $\{n_k\}$ have autocorrelation $R_n(m) = 2N_0 g_m$, having denoted by N_0 the one-sided noise power spectral density. By representing the received signal onto an orthonormal basis and denoting by $\mathbf{r}$ its vector representation, as shown in [1], it is possible to express[2]

$$p(\mathbf{r}|\mathbf{a}) \propto \prod_k G_k(a_k, a_{k-1}, \ldots, a_{k-L}) \tag{4}$$

where

$$
\begin{aligned}
&G_k(a_k, a_{k-1}, \ldots, a_{k-L}) \\
&= \exp\left\{ \frac{1}{N_0} \mathrm{Re}\left[x_k a_k^* - \frac{1}{2}|a_k|^2 g_0 - \sum_{\ell=1}^{L} a_k^* a_{k-\ell} g_\ell \right] \right\}.
\end{aligned}
\tag{5}
$$

For notational convenience, we omitted the dependence of G_k on x_k. Notice that the functions $G_k(a_k, a_{k-1}, \ldots, a_{k-L})$ are not pdfs (nor proportional to pdfs). The MAP *sequence* detection, based on the following decision rule

$$
\begin{aligned}
\hat{\mathbf{a}} &= \arg\max_{\mathbf{a}} P(\mathbf{a}|\mathbf{r}) \\
&= \arg\max_{\mathbf{a}} p(\mathbf{r}|\mathbf{a})P(\mathbf{a}) = \arg\max_{\mathbf{a}}[\ln p(\mathbf{r}|\mathbf{a}) + \ln P(\mathbf{a})] \\
&= \arg\max_{\mathbf{a}} \sum_k [\ln G_k(a_k, a_{k-1}, \ldots, a_{k-L}) + \ln P(a_k)] \\
&= \arg\max_{\mathbf{a}} \sum_k \mathrm{Re}\left[x_k a_k^* - \frac{1}{2}|a_k|^2 g_0 - \sum_{\ell=1}^{L} a_k^* a_{k-\ell} g_\ell \right] \\
&\quad + N_0 \ln P(a_k)
\end{aligned}
\tag{6}
$$

can be implemented by using the VA working on a trellis whose state is defined as $\sigma_k = (a_{k-1}, \ldots, a_{k-L})$ and by using the branch metrics

$$
\begin{aligned}
\lambda_k(a_k, \sigma_k) &= \mathrm{Re}\left[x_k a_k^* - \frac{1}{2}|a_k|^2 g_0 - \sum_{\ell=1}^{L} a_k^* a_{k-\ell} g_\ell \right] \\
&\quad + N_0 \ln P(a_k).
\end{aligned}
\tag{7}
$$

The number of trellis states is $S = M^L$. Hence, the integer parameter L represents the channel memory.

The Forney approach [2] is based on the output $\{y_k\}$ of a WMF which can be expressed as

$$y_k = \sum_{\ell=0}^{L} f_\ell a_{k-\ell} + w_k \tag{8}$$

where $\{f_\ell\}_{\ell=0}^{L}$ is the equivalent discrete-time white noise channel impulse response[3] and now the noise samples $\{w_k\}$ represent a discrete-time white Gaussian noise process. Hence, it is [2]

$$p(\mathbf{y}|\mathbf{a}) = \prod_k p(y_k|\mathbf{a}) \tag{9}$$

where

$$p(y_k|\mathbf{a}) = p(y_k|a_k, \sigma_k) \propto \exp\left\{ -\frac{1}{2N_0}\left| y_k - \sum_{\ell=0}^{L} f_\ell a_{k-\ell} \right|^2 \right\}. \tag{10}$$

Since the sequence $\{y_k\}$ represents a sufficient statistic [2], the MAP *sequence* detection strategy can be equivalently expressed as

$$
\begin{aligned}
\hat{\mathbf{a}} &= \arg\max_{\mathbf{a}} P(\mathbf{a}|\mathbf{y}) \\
&= \arg\max_{\mathbf{a}} p(\mathbf{y}|\mathbf{a})P(\mathbf{a}) = \arg\max_{\mathbf{a}}[\ln p(\mathbf{y}|\mathbf{a}) + \ln P(\mathbf{a})] \\
&= \arg\max_{\mathbf{a}} \sum_k \left\{ \left| y_k - \sum_{\ell=0}^{L} f_\ell a_{k-\ell} \right|^2 + 2N_0 \ln P(a_k) \right\}
\end{aligned}
\tag{11}
$$

and can be implemented by means of the VA working on the same trellis previously defined, with branch metrics

$$\lambda_k(a_k, \sigma_k) = \left| y_k - \sum_{\ell=0}^{L} f_\ell a_{k-\ell} \right|^2 + 2N_0 \ln P(a_k). \tag{12}$$

These two approaches for MAP *sequence* detection are exactly equivalent in terms of performance (since they are two alternative formulation of the same strategy and no approximations are involved), and trellis complexity. The only difference is in the expression of the VA branch metrics. For an in deep analysis of the computational advantages of the branch metrics (7), see the relevant discussion in [1]. In addition, the Ungerboeck approach does not require the implementation of the whitening filter necessary to obtain the sufficient statistics (8).

III. MAP SYMBOL DETECTION BASED ON THE UNGERBOECK APPROACH

The MAP *symbol* detection strategy is based on the maximization of the a posteriori probabilities (APPs) of the transmitted symbols given the received signal (or a sufficient statistics):

$$\hat{a}_k = \arg\max_{a_k} P(a_k|\mathbf{r}) = \arg\max_{a_k} P(a_k|\mathbf{y}). \tag{13}$$

All the well-known materialization of this strategy (e.g., see [6]–[8]) are based on the Forney observation model. In fact, they exploit the property that the sample y_k depends on symbols $a_k, a_{k-1}, \ldots, a_{k-L}$ only and that Gaussian noise is white. This property does not hold for samples x_k and, in fact, as previously mentioned, the factorization (4) does not involve pdfs. Hence, the probabilistic derivation usually employed to derive MAP *symbol* detection algorithms [6]–[8] cannot be extended to the Ungerboeck observation model.

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[2]In this letter, we use extensively the proportionality relationship $f \propto g$, indicating that $f = ag$ for some real constant a. In fact, a real constant is irrelevant in the maximization to be performed to implement MAP *sequence* detection whereas on the other hand, the SPA is defined up to scaling its messages by positive factors, independent of the variables represented in the graph.

[3]The coefficients $\{f_\ell\}$ can be derived from coefficients $\{g_\ell\}$ by means of a spectral factorization procedure.
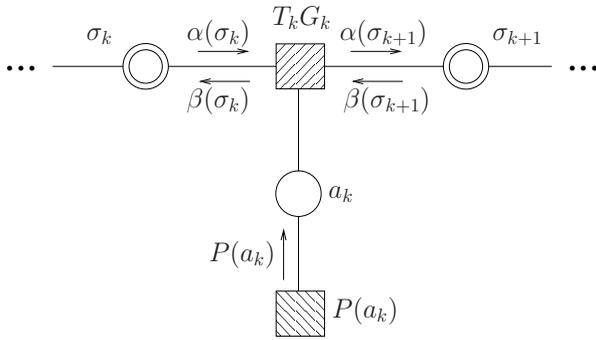
Fig. 1.   Factor graph corresponding to eqn. (14).

A MAP *symbol* detection algorithm for the Ungerboeck observation model can, however, be obtained by using the FG/SPA framework. We construct the FG corresponding to the joint APP of the transmitted symbols and the trellis states given the received signal (the same used for MAP *sequence* detection) and let the SPA compute the marginal APPs $P(a_k|\mathbf{r})$ necessary to implement MAP *symbol* detection. To obtain this result, we use the property that a FG represents the factorization of a pmf as the product of local functions that are not necessarily pmfs or pdfs [9].

The joint APP of the transmitted symbols and the trellis states $\boldsymbol{\sigma} = \{\sigma_k\}$ given the received signal can be expressed as

$$P(\mathbf{a}, \boldsymbol{\sigma}|\mathbf{r}) \propto p(\mathbf{r}|\mathbf{a})P(\boldsymbol{\sigma}|\mathbf{a})P(\mathbf{a})$$
$$= \prod_k G_k(a_k, \sigma_k)T_k(a_k, \sigma_k, \sigma_{k+1})P(a_k) \quad (14)$$

where the function $T_k(a_k, \sigma_k, \sigma_{k+1})$ is the trellis indicator function equal to 1 if a_k, σ_k, and σ_{k+1} satisfy the trellis definition and to zero otherwise [9]. In fact, since $\mathbf{a}$ and $\boldsymbol{\sigma}$ are in a one-to-one correspondence, the probability $P(\boldsymbol{\sigma}|\mathbf{a})$ is equal to 1 if all the trellis constraints are verified and to zero otherwise.

The Wiberg-type graph [9], [10] corresponding to the joint pmf in (14) is sketched in Fig. 1, where the hidden variables $\{\sigma_k\}$ have been explicitly represented. With reference to the messages in the figure, by applying the updating rules of the SPA, messages $\alpha(\sigma_k)$ and $\beta(\sigma_k)$ can be recursively computed by means of the following forward and backward recursions:

$$\alpha(\sigma_{k+1}) = \sum_{\sim\{\sigma_{k+1}\}} \alpha(\sigma_k)T_k(a_k, \sigma_k, \sigma_{k+1})G_k(a_k, \sigma_k)P(a_k)$$
$$(15)$$

$$\beta(\sigma_k) = \sum_{\sim\{\sigma_k\}} \beta(\sigma_{k+1})T_k(a_k, \sigma_k, \sigma_{k+1})G_k(a_k, \sigma_k)P(a_k)$$
$$(16)$$

whose initial conditions can be easily derived. Finally, the marginal symbol APPs can be computed by means of the following completion

$$P(a_k|\mathbf{r}) =$$
$$P(a_k)\sum_{\sim\{a_k\}} \alpha(\sigma_k)\beta(\sigma_{k+1})T_k(a_k, \sigma_k, \sigma_{k+1})G_k(a_k, \sigma_k).$$
$$(17)$$

Since the employed FG is cycle-free, the marginal APPs computed by the SPA are *exact*. Therefore, the derived algorithm is an alternative *optimal* implementation of the MAP *symbol* detection strategy. As such, the performance is the same that can be obtained, for example, by using the BCJR algorithm based on the Forney model, and for this reason, no simulation results are reported here. The structure of the algorithm is also the same. In fact, we have a forward and a backward recursion and a final completion, all working on the same trellis of the original BCJR algorithm. The only difference is that the term $G_k(a_k, \sigma_k)$ in (15), (16), and (17), is replaced, in the classical BCJR algorithm, by $p(y_k|a_k, \sigma_k)$ given in (10) [6], [9]. Hence the difference is in the expression of the branch metrics only, as for MAP *sequence* detection. As a consequence, for a complexity comparison between the proposed algorithm and the classical BCJR algorithm based on the Forney model, the same considerations mentioned in the previous section still hold.

IV. CONCLUSIONS

In this letter, a forward-backward algorithm which implements optimal MAP *symbol* detection has been derived for ISI channels and the Ungerboeck observation model. The algorithm is similar in structure to the well-known BCJR algorithm. As for MAP *sequence* detection, the Ungerboeck and Forney observation models are exactly equivalent for MAP *symbol* detection also.

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