On the Nonlinear Reference Phase in Regular Perturbation Models

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***Abstract*—We provide an analytical expression of the nonlinear phase induced by the Kerr effect that highlights its dependence on the modulation format and on the link parameters. We show that such a nonlinear phase must be used in regular-perturbation models in order to minimize their modeling error. We detail the case of dispersion-managed systems as an application example.**

I. Introduction

Any electric field propagating into an optical fiber experiences a phase shift induced by the nonlinear Kerr effect. Tracking such a phase is mandatory to make correct decisions on a linearly modulated digital signal at the receiver side. This is of concern both in a real setup as well as in modeling. While in the first case the job is efficiently performed by a phase estimator, in modeling one needs a theoretical expression for such a phase to minimize modeling errors. For instance, in the nonlinearity compensation algorithm of [2] the authors discovered the importance of such a phase, whose value was identified by a systematic search; in several Gaussian noise (GN) and enhanced GN models (EGN) [3]–[5], [7] a fake *average,* modulation format independent, phase is used for simplicity, while it was shown in [12] that modulation format does play a role in setting the average phase; the naive regular perturbation (RP) model was shown to badly fail when not referring the perturbation to a reference system rotated by the average nonlinear phase, i.e., the one where perturbation is as small as possible: such a corrected RP was called enhanced RP (eRP) [1].

The problem of the average phase reference in RP modeling can be removed by using a logarithmic perturbation (LP) [8]. However, in DU links the LP model yields log-normal statistics for the received perturbation [9], which do not match with the normal distribution observed in experiments/simulations. Alternatively, a mixed additive-multiplicative model is possible [10], however with performance that is expected to be similar to eRP [11]. A similar idea was used in [6].

In this paper we provide a rigorous evaluation of the average nonlinear phase, which accounts for the modulation format dependence, and show its use in an accurate RP model.

II. Numerical Results

By following the footsteps of [12], in the Appendix we prove the main result of this paper, namely that a linearly modulated digital signal in single polarization transmission experiences the following average nonlinear phase due to the Kerr effect:

$$\Phi_{\mathrm{NL}} \simeq \Phi_{\mathrm{GN}} \underbrace{-\frac{8}{9}\frac{\gamma\kappa_{2;2}}{PT}\int_0^L G(0,z)\int_{-\infty}^{\infty}|p(z,t)|^4\,\mathrm{d}t\mathrm{d}z}_{\Delta\Phi} \quad (1)$$

where: $\Phi_{\mathrm{GN}} \triangleq -\frac{8}{9}\gamma 2PL_{\mathrm{eff}}N$ is the (modulation-independent) phase predicted by the GN model [5], with γ nonlinear coefficient, L_{eff} fiber effective length, N number of spans, P power; $\kappa_{2;2}$ is the second order cumulant of the modulation format [12] (e.g., $\kappa_{2;2} = -1$ for quadrature phase shift keying (QPSK)); $p(z,t)$ is the supporting pulse, of energy PT with T symbol time, distorted by dispersion up to coordinate z; $G(0,z)$ is the net power gain up to z. Eq. (1) is an approximation because of the RP assumption, see Appendix.

Despite Φ_{NL} was derived in the single polarization case, in polarization division multiplexing (PDM) one just has to substitute the factor 2 in Φ_{GN} with a factor $3/2$, while the phase correcting term $\Delta\Phi$ in (1) must be weighted by $1/2$. On the other hand, in wavelength division multiplexing (WDM) Φ_{GN} must be multiplied by the number of channels, while $\Delta\Phi$ remains unchanged.

Since $\kappa_{2;2}$ is negative for all modulation formats of interest, a first implication of (1) is that $\Delta\Phi$ is always a positive correcting term. Please note that $\int |p|^4\,dt$ is *not* invariant along propagation despite the fact that a unitary transformation like dispersion makes $\int |p|^2\,dt$ invariant.

The main goal of this section is to provide numerical checks of the usefulness of eq. (1). Term $\Delta\Phi$ was neglected in GN/EGN models like [3]–[5], [7]. To have a first feeling of its importance we simulated a basic optical system based on a concatenation of a linear step L, nonlinear step N, and inverse of linear step, thus forming LNL^{-1}. Note that the RP nonlinear interference (NLI) of a true system is nothing but a linear combination of such LNL^{-1} systems, each representing RP propagation up to a specific coordinate [12].

The average nonlinear phase is shown in Fig. 1 vs. cumulated dispersion in block L. Symbols refer to split step Fourier method (SSFM) simulations, while solid line to eq. (1). We observe that while the phase used by the GN model is independent of the dispersion, the true phase does show a dependence, with a maximum mismatch around zero cumulated dispersion. The corresponding normalized NLI variance a_{NL}(NLI variance $\sigma^2_{\mathrm{NLI}} \triangleq a_{\mathrm{NL}}P^3$, where P is power) obtained by the EGN with the correct phase Φ_{NL} [12] (line) or by SSFM simulations is reported in the same figure (bottom). The numerical procedure to get NLI is summarized in Fig. 2 (top) and will be detailed later. We note a strong correlation between a_{NL} and Φ_{NL}, as well as the excellent match of EGN that uses the right phase Φ_{NL} with SSFM simulations. The EGN model with Φ_{GN} (same as [12, Fig. 7]) is indeed less accurate, while the pure GN model totally misses the dependence on the dispersion.

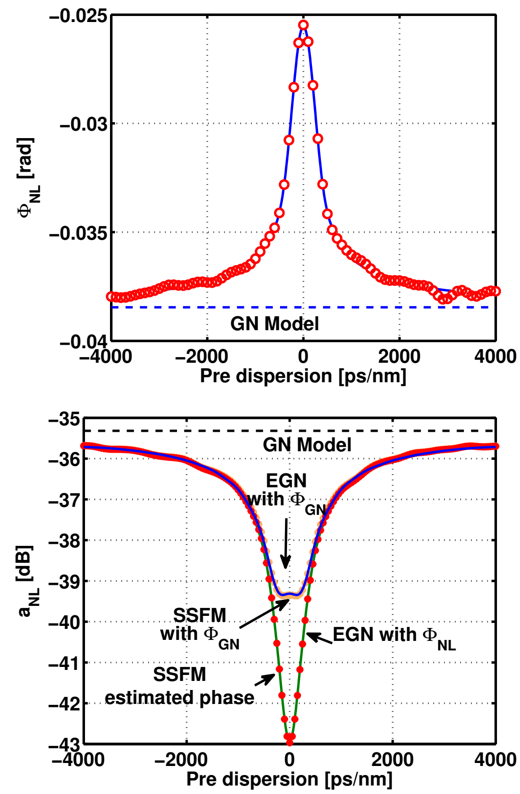

Fig. 1. (Top): Average Nonlinear phase cumulated along a concatenation of a linear step L (cumulating the pre dispersion reported on the X-axis), a nonlinear step and finally L^{-1}. Such a link corresponds to a generic branch of the RP perturbation [12]. Symbols: SSFM simulations. Solid line: eq. (1). Dashed line: GN model prediction with Gaussian distributed transmitted signal. Bottom: Corresponding normalized NLI variance ($\sigma^2_{\mathrm{NLI}} \triangleq a_{\mathrm{NL}} P^3$, P: power) [12]. Power: -1 dBm.

One more check of the importance of using the correct phase reference in RP modeling is the following. We simulated a 15 channel PDM-QPSK system with supporting sinc pulses modulated at symbol rate of 32 Gbaud with a channel spacing of 37.5 GHz. Optical link was 35×100 km long with residual dispersion per span (RDPS) of 30 ps/nm (DM30) or DU for a length of 4500 km. In the DM30 case before transmission we added a pre-compensation of -480 ps/nm. Transmission fibers had dispersion D=17 ps/nm/km, attenuation 0.2 dB/km, nonlinear coefficient $\gamma = 1.3$ 1/W/km. All amplified spontaneous emission (ASE) noise was loaded at the receiver, for an equivalent noise figure of 6 dB/amplifier. After the optical link, a post-compensating fiber recovered all the previously cumulated dispersion.

The NLI of the received central channel was extracted by using a reference phase Φ equal either to Φ_{NL} or just Φ_{GN}, as

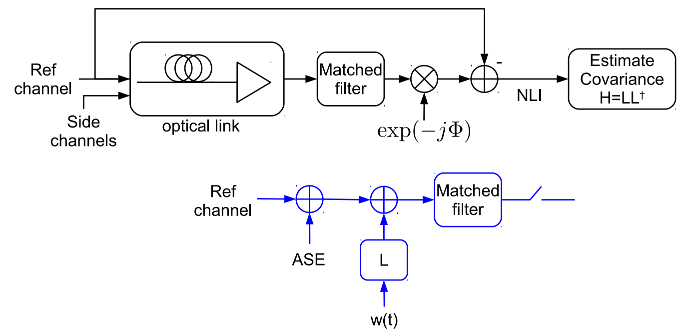

Fig. 2. Top: block diagram of the NLI extraction. Bottom: Numerical emulation of a Gaussian noise having the same covariance matrix of the NLI. Carrier phase Φ equal to either Φ_{NL} or Φ_{GN} depending on the case. w(t) is zero average, unit variance, Gaussian noise.

illustrated in Fig. 2 (top). Of such a NLI we then estimated the covariance matrix H between real/imaginary components and then its Cholesky decomposition $H = LL^T$ was derived, with T indicating transpose.

The NLI was then claimed to be a signal-independent additive Gaussian noise, as postulated by GN models [4], [5], [7], [12], and then emulated by the additive noise channel in Fig. 2 (bottom). The resulting Q-factor, estimated over random patterns of 4096 symbols each by the noise loading method, is reported in Fig. 3. For comparison, in Fig. 3 we also report the SSFM simulation of the true link. We note that by recovering only the GN phase Φ_{GN} the Q-factor is underestimated. On the other hand, by recovering the more accurate Φ_{NL}, the performance better fits with the SSFM around the best power, in agreement with the assumptions of the perturbative model. The differences are less evident in the DU case.

We next move to analyze the behavior of the nonlinear phase with system parameters. In Fig. 4 we compare eq. (1), reported by lines, with SSFM simulations (symbols) for the same optical link as in Fig. 3. RDPS is variable and reported in normalized units, i.e., $\xi \triangleq \mathrm{RDPS}/(Dz_A)$, with z_A span length. We note that all curves are monotone and saturate for $\xi \to 1$, such that Φ_{GN} is a lower bound to the unwrapped true phase. Please note that, since a phase must be read modulo 2π, the importance of $\Delta\Phi$ has to be compared with 2π and not with Φ_{GN}. Note that the 15 channel WDM curve is a rigid shift of the corresponding single channel one. Reason is that the phase correction term $\Delta\Phi$ is actually a self correcting term for each channel, whatever the channel count of the WDM, as discussed in the Appendix.

In Fig. 5 we report the phase correction term $\Delta\Phi$ versus ξ for different span numbers N. We note that the dependence on N is negligible close to the DU limit ($\xi \to 1$), while in the opposite limit it does play a role, in agreement with common wisdom that in fully compensated optical links the nonlinear phase grows linearly with the distance. In the same figure we also report (dashed lines) the approximate fit (8) which is seen to work for any value of ξ and can then be safely used either for a fast evaluation of Φ_{NL} or for inferring scaling laws.

In Fig. 6 we show $\Delta\Phi$ after a fixed number of spans $N = 40$

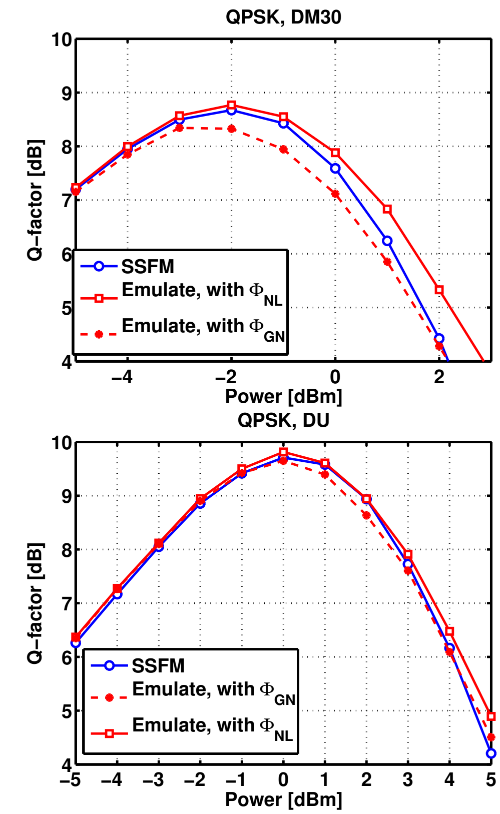

Fig. 3. Q-factor vs. power in the DM30 and DU case. DM30 is 3500 km long, while DU is 4500 km long. 15 channel PDM-QPSK system at 32 Gbaud, channel spacing 37.5 GHz.

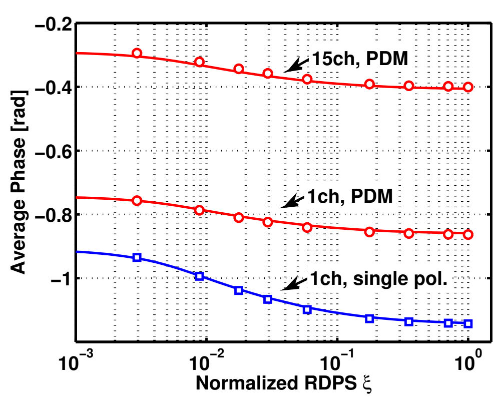

Fig. 4. Kerr induced average phase. RDPS: residual dispersion per span (DU link: $\xi = 1$. full DM link: $\xi = 0$). Symbols: Simulations. Solid line: eq. (1). Theory and numerical results confirm that conversion from single polarization to PDM is just a linear operation, see discussion after (1).

but by testing two different dispersions, 2 ps/nm/km and 17 ps/nm/km. We note that while the D=17 ps/nm/km case shows a convergence to almost zero for $\xi \to 1$, a gap remains in the D=2 ps/nm/km case, an indication that, when fiber dispersion is small, the correct reference phase system plays a role even in DU links.

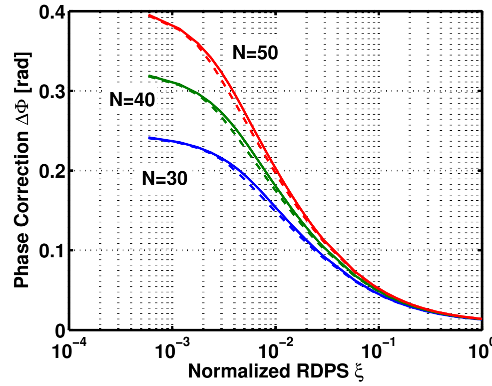

Fig. 5. Phase correction term in (1) vs. normalized RDPS at different number of span N. Solid lines: eq. (1). Dashed lines: Approximation (8). Single polarization case (in PDM multiply by $1/2$). $D = 17$ ps/nm/km, $N \times 100$ km link.

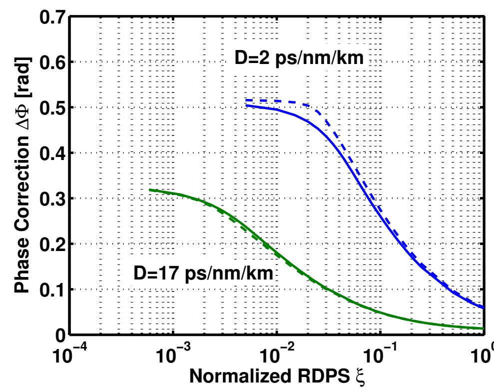

Fig. 6. Phase correction term in (1) vs. normalized RDPS at different dispersion D. Solid lines: eq. (1). Dashed lines: Approximation (8). Single polarization case (in PDM multiply by $1/2$). $N = 40$ spans.

III. Conclusions

We derived an analytical expression of the average phase induced by the nonlinear Kerr effect, which accounts for the modulation format. We showed that the correct phase is the one predicted by the GN model plus a correcting term, always positive for the modulation formats of interest.

Appendix

Average Phase Derivation

Target is estimating the average nonlinear phase experienced by the transmitted linear digital modulated signal:

$$U(t) = \sum_k a_k p(0, t - kT)$$

where a_k are zero mean complex symbols with at least a 4-fold rotational symmetry. The optical link is a generic periodic DM link. Given the RP1 approximation of the received field:

$$U_{\mathrm{RP}}(t) = U_0 + U_1$$

with $U_{0,1}$ unperturbed/perturbed solutions [4], [12], respectively, the maximum likelihood estimator of the nonlinear induced phase, by assuming U_1 Gaussian distributed, is [13]:

$$\Phi_{\text{NL}} = \arg\left[\int_{T_0} U_{\text{RP}}(t)U_0^*(t)\text{d}t\right] \tag{2}$$

where T_0 is the observation window. For a small nonlinear effect the phase can be safely linearly related to the imaginary component:

$$\Phi_{\text{NL}} \simeq \frac{\Im\left[\int_{T_0} U_1(t)U_0^*(t)\text{d}t\right]}{\int_{T_0}|U_0(t)|^2\,\text{d}t}$$

in agreement with the RP1 idea that powers of γ higher than 1 have negligible effect. Now let $T_0 \to \infty$. By invoking ergodicity for the electric field we can take the expected value of the integrands, obtaining:

$$\Phi_{\text{NL}} \simeq \frac{\Im\left[R^{(0)}_{U_1U_0}(0)\right]}{\Re\left[R^{(0)}_{U_0U_0}(0)\right]} \tag{3}$$

where $R^{(n)}_{XY}(\tau)$ is the cyclic cross-correlation of $X(t)$ and $Y(t)$ at cycle frequency n/T:

$$R^{(n)}_{XY}(\tau) \triangleq \frac{1}{T}\int_T E\left[X(t+\tau)Y^*(t)\right]e^{-j2\pi n\frac{t}{T}}\text{d}t \tag{4}$$

with $E[.]$ expectation. With perfect dispersion compensation at the end of the link, fields $U_{0,1}$ can be written as [4], [12]:

$$\begin{aligned} U_0(t) &= U(t) \\ U_1(t) &= -j\gamma\int_0^L h_{z0}(t)\otimes V(z,t)\text{d}z \end{aligned} \tag{5}$$

where L is the link length and:

$$\begin{aligned} V(z,t) &\triangleq |U_0(z,t)|^2\,U_0(z,t) \\ U_0(z,t) &\triangleq h_{0z}(t)\otimes U(t) \\ \tilde{h}_{sz}(\omega) &\triangleq \int_{-\infty}^{\infty} h_{sz}(t)e^{-j\omega t}dt = \sqrt{G(s,z)}e^{-j\frac{\int_s^z \beta_2(x)\text{d}x}{2}\omega^2} \end{aligned}$$

with $\otimes$ indicating time-convolution. $h_{sz}(t)$ is a filter accounting for linear effects from s to z, like the net gain G as well as dispersion β_2. By substituting (5) into (4) we have [12]:

$$R^{(0)}_{U_1U_0}(\tau) = -j\gamma\int_0^L h_{s0}(\tau)\otimes h^*_{s0}(-\tau)\otimes R^{(0)}_{VU_0}(s,\tau)\text{d}s\ .$$

As in [12], by introducing $Y_k \triangleq a_k g_k \triangleq a_k p(z, t-kT)$, such that $p(z,t) = h_{0z}(t)\otimes p(0,t)$, we have:

$$\begin{aligned} R^{(0)}_{VU_0}(t,\tau) &= \sum_{k,n,l,m} E\left[Y_kY_n^*Y_lY_m^*\right] \\ &= 2\kappa_{1;1}^2\sum_k |g_k|^2\sum_n |g_n|^2 + \kappa_{2;2}\sum_k |g_k|^4 \\ R^{(0)}_{U_0U_0}(t,\tau) &= \sum_{k,n} E\left[Y_kY_n^*\right] = \kappa_{1;1}\sum_k |g_k|^2 \end{aligned} \tag{6}$$

where $\kappa_{n;n}$ is the n-th order cumulant of the information symbols a_k [12]. By exploiting the properties of cyclostationary signals, we have:

$$\frac{1}{T}\int_T \sum_k |g_k|^n\,\text{d}t = \frac{1}{T}\int_{-\infty}^{\infty} |p(z,t)|^n\,\text{d}t\,. \tag{7}$$

Collecting together (3), (6) and (7) we finally get (1). Note that in WDM with independent channels it is impossible to find a set of four equal symbols in (6), so that cross channel induced average phase is exactly captured by the GN model.

In the simplified, yet relevant, case of sinc pulses and by using the stationary phase approximation, we found the following excellent approximation:

$$\begin{aligned} \Delta\Phi \simeq -\frac{8}{9}\gamma P\kappa_{2;2}\sum_{k=1}^{N}\Bigg[&\frac{2}{3}\frac{1-e^{-\alpha M(k)}}{\alpha} \\ &-\frac{\log\left(1+\frac{2}{\alpha(M(k)+(k-1)s_{\text{in}}+s_{\text{p}})}\right)}{2\pi|\beta_2|R^2}\frac{e^{-\alpha M(k)}}{2}\Bigg] \end{aligned} \tag{8}$$

where $s_{\text{in}} \triangleq \text{RDPS}/D$, $s_{\text{p}} \triangleq D_{\text{pre}}/D$, D_{pre} being the cumulated dispersion into a pre-compensating fiber before transmission, and $M(k) \triangleq \max\left(0,\ \frac{3}{4\pi|\beta_2|R^2} - s_{\text{p}} - (k-1)s_{\text{in}}\right)$.

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