On the Information Rate and Repeat-Accumulate Code Design for Phase Noise Channels

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***Abstract*—We investigate the information rate of channels affected by phase noise, aiming at predicting the ultimate performance limits in this scenario. Moreover, a closed-form upper bound is also derived for phase shift keying (PSK) modulations. Finally, we consider the design of nonsystematic irregular repeat-accumulate (RA) codes for this channel trying to give new insights on the codes to be employed for such an application.**

***Index Terms*—Differential encoding, iterative detection and decoding, detection and decoding in the presence of phase noise.**

I. INTRODUCTION

IN SOME communication links, the adoption of low-cost transmit and/or receive oscillators makes the phase noise one of the major impairments. An example is represented by the next generation digital video broadcasting satellite systems (DVB-S2) [1], where unexpensive low-noise blocks in the outdoor units and tuners in the indoor units introduce a strong phase noise. Similarly, laser's phase noise strongly degrade the performance of the upcoming 100 Gbps long-haul optical coherent systems [2].

In the literature, this phase noise is commonly modeled as a Wiener process, although more accurate models have been recently proposed for consumer-grade equipments to be employed in DVB-S2 systems [3]. In this paper, however, the Wiener model will be considered mainly because it is characterized by a single parameter which allows effective tuning of its strength, and also because a receiver designed for the Wiener phase noise model also performs well for a more accurate phase noise model, suggesting that there is no significant practical difference between the simplified and the more accurate models [4].

For channels affected by phase noise, a lot of papers have addressed in detail the problem of detection (as examples, see [4]–[7] and references therein) but nothing has been said, to the best of our knowledge, from an information theoretic point of view if we except the case of a constant channel phase offset or error [8]–[10] and the case of ideal interleaving (and thus, uncorrelated phase noise samples) [11]. Our aim here is to analyze, through the computation of the information rate (IR), i.e., the average mutual information when the channel inputs are independent and uniformly distributed (i.u.d.) random variables, the ultimate performance loss due to the presence of phase noise. In addition, by designing, through EXtrinsic Information Transfer (EXIT) charts [12], specific repeat-accumulate (RA) codes for such an application, we discuss the common practice of employing, in satellite applications, channel codes designed for memoryless channels [1].

The theoretical framework over which our investigation is based, is represented by the method, independently proposed by several authors in recent years, for the evaluation of the IR for finite-state channels [13]. Although the considered phase noise channel is not finite-state, this approach will be pursued by resorting to a proper auxiliary channel and deriving lower bounds on the information rate achievable by a maximum-likelihood decoder for the auxiliary channel.

Regarding code design for channels with phase noise, we would like to mention the alternative approach described in [14]. In that paper, low-density parity-check (LDPC) codes over rings are designed by dividing the codewords into sub-blocks of adjacent symbols under the assumption that the phase variations over each of them are small. Two classes of check nodes are then created: the "global check nodes", spread across many sub-blocks, that converge irrespective of possible rotations of a multiple of the rotational invariance angle for the employed constellation, and "local check nodes" inserted to resolve the phase ambiguity on each sub-block. At the receive end, the joint detection/decoding process is modified accordingly. Global check nodes are used first and different phase estimates are produced, one for each sub-block. The sub-block phase ambiguities are then solved by exploiting local check nodes. Although very interesting from a conceptual viewpoint, this approach has the following drawbacks. First of all, as the authors explicitly admit, it is not able to tackle large amounts of phase noise, such as those considered in this paper. In addition, the stronger the phase noise, the lower the optimal sub-block size and hence the larger the decoding complexity. Finally, the detection/decoding procedure has an intrinsic loss of a few tenth of dBs (also when the channel phase is constant) since it does not include the local check nodes during the first stage and for the degraded performance of the turbo phase estimator whith reduced sub-block size. On the contrary, the codes described in this paper, for which phase ambiguity is solved through the intrinsic differential encoding, do not require an ad-hoc decoding procedure. Hence, the algorithms in [7], with a practically optimal performance irrespective of the amount of phase noise, can be employed at the receiver.

The rest of the paper is organized as follows. The exact system model is described in Section II, whereas in Section III

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the algorithm in [13] is briefly reviewed and specialized for the problem at hand. In Section IV, a very simple but useful upper bound on the IR for M-ary phase shift keying (M-PSK) modulations is obtained in closed-form. In Section V, a technique based on EXIT charts [12] is pursued for the design of irregular RA code design for this scenario whereas in Section VI numerical results are presented. Finally, in Section VII some conclusions are drawn.

II. System Model

We consider the transmission of a sequence of complex modulation symbols $\mathbf{x} = \{x_k\}$, independent, identically, and uniformly distributed, belonging to an M-ary complex alphabet $\mathcal{X} = \{x^{(i)}\}_{i=0}^{M-1}$, over an additive white Gaussian noise (AWGN) channel affected by an unknown time-varying channel phase. Considering a linear modulation at the transmitter side and assuming that one sample per symbol is adequate (as in the absence of strong phase variations within a few symbol periods), if transmit and receive filters are such that there is absence of intersymbol interference, we have the following observation model

$$y_k = x_k e^{j\theta_k} + w_k \tag{1}$$

where noise samples $\mathbf{w} = \{w_k\}$ are independent and identically distributed, complex, circularly symmetric, Gaussian random variables, each with mean zero and variance equal to $2\sigma^2$. We will denote the sequence of received samples by $\mathbf{y} = \{y_k\}$. Given a generic sequence $\mathbf{v} = \{v_k\}$, we will also denote by $\mathbf{v}^n = \{v_k\}_{k=1}^n$ the sequence composed by its first n elements.

A common model for the phase noise process $\{\theta_k\}$ is the random-walk (Wiener) model described by

$$\theta_{k+1} = \theta_k + \Delta_k \tag{2}$$

where $\{\Delta_k\}$ is a discrete-time white real Gaussian process with mean zero and variance σ_Δ^2, and θ_0 is uniformly distributed in the interval $[0, 2\pi)$. Hence, it follows that

$$p(\theta_k|\theta_{k-1}, \theta_{k-2}, \ldots, \theta_0) = p(\theta_k|\theta_{k-1}) = p_\Delta(\theta_k - \theta_{k-1}) \tag{3}$$

where we define $p_\Delta(\varphi)$ as the probability density function (pdf) of the increment $\Delta_k \bmod 2\pi$, i.e.,

$$p_\Delta(\varphi) = \sum_{\ell=-\infty}^{\infty} g\left(0, \sigma_\Delta^2; \varphi - \ell 2\pi\right) \quad , \quad \varphi \in [0, 2\pi) \tag{4}$$

having denoted by $g(\eta, \rho^2; x)$ a real Gaussian pdf with mean η, variance ρ^2, and argument x. The sequence of phase increments $\{\Delta_k\}$ is supposed unknown to both transmitter and receiver and statistically independent of $\mathbf{x}$ and $\mathbf{w}$.

We also consider the following finite-state auxiliary channel model in which the channel phases $\boldsymbol{\theta} = \{\theta_k\}$ belong to a finite set such that

$$\theta_k \in \left\{\frac{2\pi}{L}\ell\right\}_{\ell=0}^{L-1} \tag{5}$$

the larger the number of discretization levels L, the better the approximation. Taking into account (2) and (4), it is possible to find the transition probabilities of the discretized model as

$$\begin{aligned} P_{i,j} &= P\left(\theta_{k+1} = \frac{2\pi}{L}j \Big| \theta_k = \frac{2\pi}{L}i\right) \\ &= \int_{(j-i-\frac{1}{2})\frac{2\pi}{L}}^{(j-i+\frac{1}{2})\frac{2\pi}{L}} \sum_{\ell=-\infty}^{\infty} g\left(0, \sigma_\Delta^2; \varphi - \ell 2\pi\right) d\varphi \\ &= \sum_{\ell=-\infty}^{\infty} \left[Q\left((j-i-\frac{1}{2})\frac{2\pi}{L\sigma_\Delta} - \frac{2\pi\ell}{\sigma_\Delta}\right) \right. \\ &\quad \left. - Q\left((j-i+\frac{1}{2})\frac{2\pi}{L\sigma_\Delta} - \frac{2\pi\ell}{\sigma_\Delta}\right)\right] \end{aligned} \tag{6}$$

where $Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$ is the Gaussian Q function.

III. Evaluation of the Information Rate

A lower bound for the information rate of the channel at hand is the information rate achievable by a receiver designed for the auxiliary channel with discretized phase, defined by (1) and (5), when the actual channel is the original one with Wiener phase noise. We also expect that the larger the value of L, the tighter this lower bound. This issue, which is an instance of *mismatched decoding* [15], cannot be addressed in closed form, but can be solved by means of the simulation-based method described in [13], which only requires the existence of an algorithm for exact maximum a posteriori (MAP) symbol detection over the auxiliary channel. For the considered auxiliary channel, MAP symbol detection is an instance of the well known Bahl-Cocke-Jelinek-Raviv (BCJR) algorithm [16] working on a trellis with L states.

The achievable IR for the mismatched receiver can be evaluated as

$$\begin{aligned} I(\mathbf{x}; \mathbf{y}) &= \lim_{n\to+\infty} \frac{1}{n} I(\mathbf{x}^n; \mathbf{y}^n) \\ &= \lim_{n\to+\infty} \frac{1}{n} E\left\{\log \frac{p(\mathbf{y}^n|\mathbf{x}^n)}{p(\mathbf{y}^n)}\right\} \left[\frac{\text{bit}}{\text{symb.}}\right] \end{aligned} \tag{7}$$

where $p(\mathbf{y}^n|\mathbf{x}^n)$ and $p(\mathbf{y}^n)$ are probability density functions according to the auxiliary channel model, while the outer statistical average is with respect to the input and output sequences evaluated according to the actual channel model [13]. Both $p(\mathbf{y}^n|\mathbf{x}^n)$ and $p(\mathbf{y}^n)$ can be evaluated recursively through the forward recursion of the MAP detection algorithm matched to the auxiliary channel model [13]. Let us recall that the mismatched receiver can assure error-free transmissions when the transmission rate at the modulator input does not exceed $I(\mathbf{x}; \mathbf{y})$ bit/symbol.

IV. Upper Bound for M-PSK Modulations

It would be also desirable to find some closed-form bounds for the IR. To obtain a closed-form result, a simple hypothesis, which is however largely verified in all practical channel conditions, is to consider $\sigma_\Delta \ll 2\pi$, such that the pdf of the phase increment (4) is practically Gaussian. We need the following preliminary result:

Theorem 1: For an M-PSK modulation, in the absence of thermal noise (i.e., $\sigma = 0$), the average mutual information is a non-decreasing function of M.

Proof: Let us build a $2M$-PSK modulation with this trick: for every time epoch k the transmitted phase $\alpha_k \in$

$\left\{\frac{\pi}{M}m\right\}_{m=0}^{2M-1}$ is evaluated in this way: $\alpha_k = \alpha'_k + \frac{\pi}{M}s_k$, where $\alpha'_k \in \left\{\frac{2\pi}{M}m\right\}_{m=0}^{M-1}$ and $s_k \in \{0,1\}$ with equal probability. It is like splitting the $2M$-PSK constellation into two M-PSK constellations, where at each time epoch k the constellation is selected by the flag s_k.

Let us moreover define the sequence $\mathbf{z} = \{z_k\}$ of observed samples as $z_k = [\alpha_k + \theta_k]_{-\pi}^{\pi}$. We also define $\boldsymbol{\alpha} = \{\alpha_k\}$ and $\mathbf{s} = \{s_k\}$. Since the thermal noise is absent, this channel model is equivalent to (1), so we have to prove that

$$I(\boldsymbol{\alpha};\mathbf{z}) \geq I(\boldsymbol{\alpha};\mathbf{z}|\mathbf{s}=\mathbf{0}) \tag{8}$$

which is equivalent, in terms of entropies [17] to

$$h(\mathbf{z}) - h(\mathbf{z}|\boldsymbol{\alpha}) \geq h(\mathbf{z}|\mathbf{s}=\mathbf{0}) - h(\mathbf{z}|\boldsymbol{\alpha},\mathbf{s}=\mathbf{0})\,. \tag{9}$$

The entropy $h(\mathbf{z})$ is defined as follows

$$\begin{aligned} h(\mathbf{z}) &= \lim_{n\to+\infty}\frac{1}{n}h(\mathbf{z}^n) \\ &= -\lim_{n\to+\infty}\frac{1}{n}E[\log(p(\mathbf{z}^n))] \end{aligned} \tag{10}$$

and similarly for the conditional entropy $h(\mathbf{z}|\boldsymbol{\alpha})$ with $p(\mathbf{z}^n|\boldsymbol{\alpha}^n)$ in place of $p(\mathbf{z}^n)$.

Both terms $h(\mathbf{z}|\boldsymbol{\alpha})$ and $h(\mathbf{z}|\boldsymbol{\alpha},\mathbf{s}=\mathbf{0})$ are an average of terms of the form $h(\mathbf{z}|\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}})$ but the first one over the entire $2M$-PSK sequence space whereas the second one on the set of sequences with $\mathbf{s}=\mathbf{0}$. Clearly $h(\mathbf{z}|\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}) = h(\boldsymbol{\theta})$ is independent of the sequence $\boldsymbol{\alpha}$, and thus also $h(\mathbf{z}|\boldsymbol{\alpha}) = h(\mathbf{z}|\boldsymbol{\alpha},\mathbf{s}=\mathbf{0})$. This ends the proof, since $h(\mathbf{z}) \geq h(\mathbf{z}|\mathbf{s}=\mathbf{0})$ because conditioning reduces entropy. ■

We would like to point out that this result does not hold in the presence of thermal noise.

We can now prove the following Theorem:

Theorem 2: For any M-PSK modulation over the Wiener phase noise channel, with a parameter σ_Δ, the average mutual information is upper bounded by

$$I(\mathbf{x};\mathbf{y}) \leq \log_2\left(\frac{\sqrt{2\pi/e}}{\sigma_\Delta}\right). \tag{11}$$

Proof: By using Theorem 1 and the fact that the average mutual information is a non-decreasing function of the signal-to-noise ratio, it is sufficient to prove that (11) is valid for $\sigma = 0$ and $M \to +\infty$. This is moreover equivalent to considering a channel with an input $\boldsymbol{\alpha}$ uniformly distributed in $[-\pi,+\pi]$ transmitted over a channel such that $z_n = [\alpha_n + \theta_n]_{-\pi}^{+\pi}$. Now, by direct computation of

$$I(\boldsymbol{\alpha}^n;\mathbf{z}^n) = \int p(\boldsymbol{\alpha}^n)\int p(\mathbf{z}^n|\boldsymbol{\alpha}^n)\log_2\frac{p(\mathbf{z}^n|\boldsymbol{\alpha}^n)}{p(\mathbf{z}^n)}d\mathbf{z}^n d\boldsymbol{\alpha}^n \tag{12}$$

it can be shown that $I(\boldsymbol{\alpha}^n;\mathbf{z}^n) \leq (n-1)\log_2\left(\frac{\sqrt{2\pi/e}}{\sigma_\Delta}\right)$, from which (11) follows directly. ■

This upper bound states that very dense PSK constellations, in which $\log_2 M$ is greater than the bound, do not reach the value $\log_2 M$ even in the absence of thermal noise (i.e., for high signal-to-noise ratio values). Furthermore, it shows that, as it was expected, the maximum mutual information decreases when the speed of variation of the phase noise (σ_Δ) increases. Moreover, it will be shown that this bound is very tight, since M-PSK modulations such that $\log_2 M$ is much larger than the bound have an average mutual information that reaches the bound for large enough signal-to-noise ratio values.

A similar nice result cannot be obtained for non-equal energy signals since the information carried by the amplitude variations is not affected by the time-varying nature of the phase.

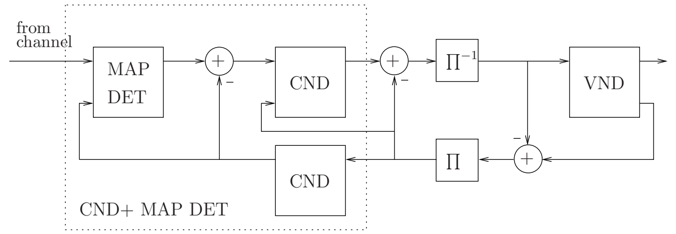

Fig. 1. Iterative receiver for a nonsystematic RA code.

V. RA Code Design

We now provide a possible solution to the problem of finding good codes for this scenario. We use EXIT charts [12] and linear programming to find the parity check degree distribution of a nonsystematic irregular RA code such that, for a given signal-to-noise ratio value, the code exhibits the largest rate. We apply a design strategy similar to that applied in [18] for a multiple-input multiple-output channel.

The iterative receiver for this case, whose scheme is shown in Fig. 1, is the serial concatenation of three soft-input soft-output (SISO) blocks: the variable node decoder (VND), the check node decoder (CND) and the MAP SISO detector for a differentially encoded modulation (MAP DET). This MAP SISO detector is based on phase quantization and is described in [7].[1] An interleaver (Π) is placed between the CND and the VND. For the optimization procedure, it is necessary to join the CND and the MAP SISO detector in a macroblock (CND+MAP DET in the figure) [18]. It is worth noticing that pilot symbols are not necessary (as it would be in the absence of differential encoding), since the detector exploits the differential nature of the transmitted data in order to avoid phase ambiguities. We use here a different notation than that of [18]. Let D_v be the number of different variable node degrees and denote these degrees by $d_{v,i}$, $i=1,\ldots,D_v$. The average variable node degree is

$$\overline{d}_v = \sum_{i=1}^{D_v} a_i d_{v,i} \tag{13}$$

where a_i is the fraction of *nodes* having degree $d_{v,i}$. Moreover, we denote by λ_i the fraction of *edges* incident to variable nodes of degree $d_{v,i}$. Clearly both the a_i and λ_i sum up to one. Similarly, we denote by D_c, $d_{c,i}$, b_i, and ρ_i the number of different check node degrees, the i-th degree, and the fractions of check nodes and edges having degree $d_{c,i}$, respectively. Since the number of edges at the VND and CND are the same, we have $\overline{d}_c = R\overline{d}_v$, R being the code rate.

[1]As specified in Section VI, we employ a number of quantization levels typical of low-complexity practical detectors.

In order to obtain a linear optimization problem, the check node distribution has to be chosen a priori. The optimization algorithm consists of finding a variable node degree distribution, namely the $d_{v,i}$ along with their λ_i (for a given value of D_v) such that the code rate R is maximized and the EXIT chart tunnel is open.

Let us consider the average mutual information from the variable node decoder, available in [18], and that going into the VND. This latter average mutual information is a cumbersome function, computable only through numerical simulations, parameterized by the thermal noise and phase noise variances (σ^2 and σ_Δ^2, respectively) and by the check node degree distribution [18]. The optimization problem at hand can be expressed as a linear programming problem, similar to those reported in [19], where the objective function to be maximized is the code rate R, which is equivalent to

$$\max_{\{\lambda_i\}_i} \sum_i \lambda_i / d_{v,i}$$

and the constraints are the open tunnel condition [18], that can be formulated as to have an increasing average mutual information exchanged by the SISO components, and

$$\sum_i \lambda_i = 1 \quad , \quad \sum_i \lambda_i / d_{v,i} \leq \overline{d}_c$$

the second ensuring a code rate not greater than one.

The optimization algorithm proceeds as follows. First of all, the check node distribution and the minimum and maximum tolerable values for the variable node degrees are chosen. Since we are considering a non-systematic RA code, the iterative procedure can start only in the presence of check nodes with degree one. Indeed, a small amount of degree-one check nodes must be inserted in order to allow the bootstrap of the decoding algorithm (the so-called *code doping* [18]). We impose a biregular structure for the check node degrees, i.e., $d_{c,1} = 1$ with $b_1 = 0.2$ and $d_{c,2} = d_c$ with $b_2 = 0.8$, where d_c is also optimized by the optimization program.

The signal-to-noise ratio is incremented step by step starting from a minimum value. For each value, several candidate values of d_c are tried, from $d_c = 2$ to $d_c = 10$, the linear programming problem is solved for each value and the distribution $\{\lambda_i\}_i$ that maximizes the code rate, as well as the rate itself, are evaluated. The value of d_c which guarantees the largest rate is chosen. It is worth noticing, however, that the linear problem could not have a valid solution (i.e., a solution for which all constraints are satisfied). This happens when the signal-to-noise ratio value is too small with respect to the modulation and the channel conditions to have a reliable communication.

VI. Numerical Results

We now present, for different modulation formats, the information rate of channels with Wiener phase noise as a function of E_b/N_0, E_b being the received signal energy per information bit and N_0 the one-sided noise power spectral density. As a matter of comparison, two curves for the AWGN channel have been added to some of the figures: the unconstrained capacity, given by the recursive equation

$$R = \log_2\left(1 + R\frac{E_b}{N_0}\right) \tag{14}$$

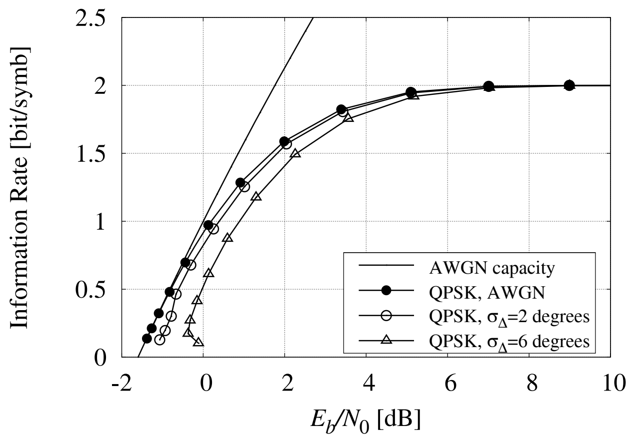

Fig. 2. Information rate for QPSK modulation.

and the information rate which instead depends on the modulation format.

In order to obtain very tight lower bounds for the information rate in the presence of phase noise, in our simulations we used $L = 1024$ discretization levels for the channel phase and sequences of length of $n = 10^5$ symbols. We indeed verified that a further increase of these values do not lead to significantly different results. This number of discretization levels is much higher than that necessary for a practical receiver, based on phase discretization, to obtain near optimal performance.

In Fig. 2, the IR for the phase noise channel is shown for a quaternary PSK (QPSK) modulation compared with capacity and IR of the AWGN channel. Two different values of σ_Δ are considered. As it can be observed, for any given information rate, there is a variable energy loss of the curves for the phase noise channels. Indeed, as expected, the loss is proportional to the variance of the phase noise increments, i.e., the larger the σ_Δ value, the larger the energy loss. For instance, at an information rate of 1 bit/symbol, the channel with $\sigma_\Delta = 2$ degrees exhibits a loss of about 0.2 dB with respect to the constrained capacity of the AWGN channel, while the loss increases to about 0.7 dB for the channel with $\sigma_\Delta = 6$ degrees. Clearly, the IR of a QPSK modulation over any channel cannot go over 2 bit/symbol. In this figure, the reader can observe a particular behavior of the information rate curve I as a function of E_b/N_0, that will be even more clear in the following figures. In fact, there exists a value of I such that, for lower values, E_b/N_0 increases. If we plot the information rate I as a function of E_S/N_0, this behavior obviously disappears since, for physical reasons, I is a non decreasing function of E_S/N_0. On the other hand, E_S is related to E_b through the equation $E_S = IE_b$. Hence, given a value of E_b/N_0, we have to solve the fixed-point equation $f(x) = x$, where $x = \frac{E_S}{N_0}$ and $f(x) = I\left(\frac{E_S}{N_0}\right)\frac{E_b}{N_0}$, that must be solved for $x \geq 0$ and $f(x) \geq 0$ (in the first quadrant). In the case of the AWGN channel, for every value of E_b/N_0 we always have just one solution (excepting the trivial one for $(x, f) = (0, 0)$) of the fixed-point equation. In the case of a phase noise channel, for some values of E_b/N_0 we have two solutions. The value of E_b/N_0 such that E_b/N_0 increases when I decreases is the

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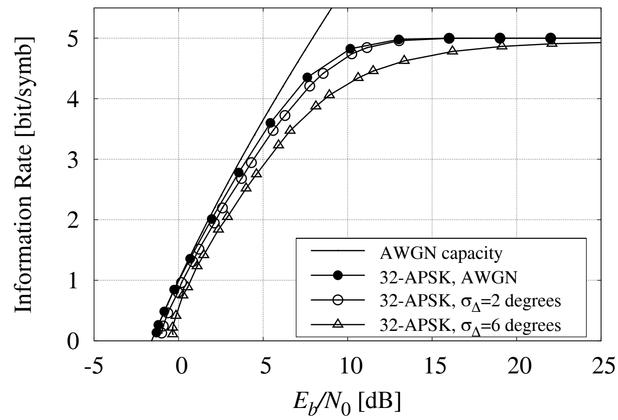

Fig. 3. Information rate for a 32-APSK modulation.

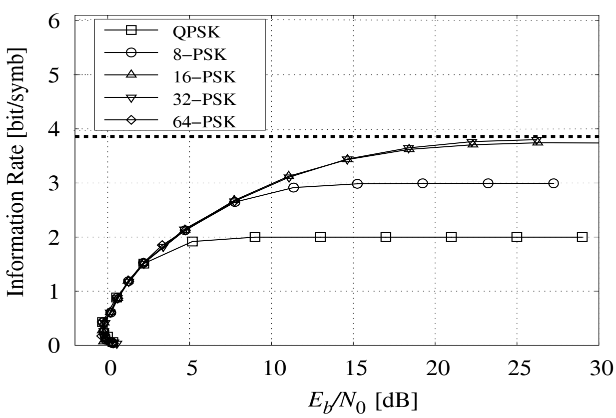

Fig. 4. Information rate for several M-PSK modulations, compared with the bound (11).

value x_0 of x such that

$$\left.\frac{df(x)}{dx}\right|_{x=x_0} = 1 \quad , \quad f(x_0) = x_0 \, .$$

Curves similar to those in Fig. 2 for QPSK are reported in Fig. 3 for an amplitude/phase shift keying (APSK) modulation with 32 symbols. APSK constellations are the composition of a set of PSK constellations with different radius. They are particularly suited for satellite communications, thanks to their robustness to channel non-linearities and thus they have been standardized for the future DVB-S2 communication systems [1]. 32-APSK is in particular built by 3 concentric PSKs (an inner 4-PSK, a medium 12-PSK and an outer 16-PSK). Due to the higher density of the considered constellation with respect to QPSK, the energy loss for a given IR from the AWGN curve is larger than in the QPSK case. For instance, at an IR of 3.5 bits/symbol, the losses are 0.6 dB and 1.3 dB for the channels with $\sigma_\Delta = 2$ degrees and $\sigma_\Delta = 6$ degrees, respectively.

In Fig. 4, the mutual information of M-PSK modulations for several different values of M are plotted together with the upper bound (11) (the horizontal line), which in the considered case of $\sigma_\Delta = 6$ degrees, becomes $I(\mathbf{x};\mathbf{y}) \leq 3.85$ bit/symbol. As it can be seen, all modulations such that $\log_2 M \geq 4$ are not able to reach their maximum value $\log_2 M$ not even for very large signal-to-noise ratio values and in practice perform identically. Moreover, the bound is very tight, since the curves of all constellations with $M \geq 16$ converge to the horizontal line for a sufficiently large E_b/N_0 value.

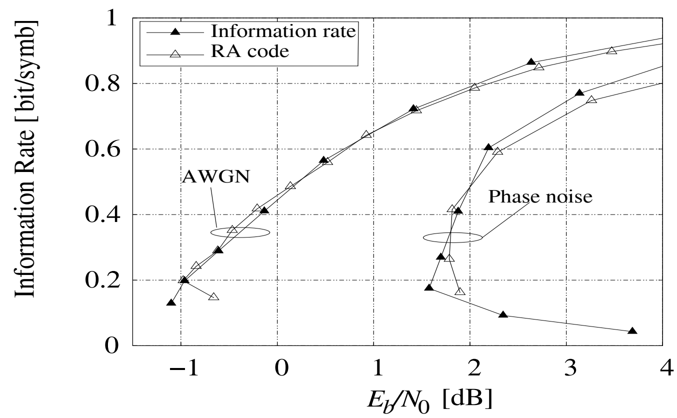

Fig. 5. Code rate compared with the information rate for a channel with a phase noise with $\sigma_\Delta = 20$ degrees.

TABLE I
BEST MODULATIONS FOR DIFFERENT SIGNAL-TO-NOISE RATIO VALUES.

E_b/N_0	Modulation
$\cdot < 0.2\,\mathrm{dB}$	16-QAM
$0.2\,\mathrm{dB} < \cdot < 10\,\mathrm{dB}$	64-QAM
$10\,\mathrm{dB} < \cdot$	256-QAM

It is well known that, on the AWGN channel, larger constellations yield better IR. On the contrary, as we said in Section IV, on the channel with both thermal and phase noise, there can be an optimal constellation, different for every considered signal-to-noise ratio value, which behaves better than any other in a given set. We take into account all the following modulations: M-PSK with $\log_2 M = 2, 3, 4, 5, 6$, M-QAM with $\log_2 M = 4, 6, 8$ and 32-APSK, on a channel with $\sigma_\Delta = 6$ degrees. The corresponding figure is not reported here since it would be not clear due to the large number of curves. However, we manually checked, for various E_b/N_0, the modulation which reaches the largest rate. The results are reported in Table I.

Some interesting observations can be drawn from these results. First of all, amplitude modulations seem to always outperform the PSKs. This can be explained by the fact that the information conveyed by the amplitude variations is not affected by the time-varying phase but by the thermal noise only. Indeed, M-QAM modulations clearly overcome the bound (11) for PSK modulations. Moreover, it can be seen that at low signal-to-noise ratio values, less dense constellations perform better.

Finally, in Fig. 5 we show the best code rate found as a function of E_b/N_0 ($E_b/N_0 = R\,E_S/N_0$, E_S being the received signal energy per information symbol), compared with the IR, for a BPSK modulation with a phase noise characterized by $\sigma_\Delta = 20$ degrees. As mentioned, at the receiver end, the MAP SISO detector described in [7] is employed with $L = 12$ phase discretization levels. This value is sufficient to give a practically optimal performance. We set the maximum variable node degree to 25 and the minimum to 3. RA codes for both the AWGN channel and the channel

TABLE II
DESIGNED CODES FOR A CHANNEL WITH PHASE NOISE AND COMPARISON WITH AWGN. TWO VALUES OF E_S/N_0 HAVE BEEN CONSIDERED.

AWGN					PN with σ_Δ =6 degrees				
$\frac{E_s}{N_0}$	d_c	$d_{v,i}$	λ_i	R	$\frac{E_s}{N_0}$	d_c	$d_{v,i}$	λ_i	R
-4 dB	3	2 8 22 23	0.28 0.34 0.36 0.02	0.395	-4 dB	3	2 7 8 25	0.24 0.13 0.22 0.41	0.37
1 dB	4	3 14 15 23	0.57 0.21 0.08 0.14	0.74	1 dB	4	3 14 15 24	0.57 0.12 0.18 0.13	0.73

affected by phase noise were designed, for different signal-to-noise ratio values. As it can be seen, for both channel models the optimization procedure we used was able to obtain codes with a theoretical threshold with only a negligible loss with respect to the constrained capacity, at least for rates not larger than 0.8. Better results for such large rates could be obtained by considering also variable nodes of degree 2, which was however excluded in our optimization because the presence of degree-2 variable nodes makes the code distance spectrum poor and thus tends to increase the floor of the resulting code [18]. Moreover, it is possible to see that the code rate exhibits a saturation, i.e., there exists a maximum value which is never exceeded. This behavior is due to the constraint on the maximum check node degree, which in turn translates to a maximum value of the rate. Bit error rate results, not shown here, demonstrate that for code lengths of a few thousands of bits, the performance is within 1 dB from the theoretical results, thus confirming the effectiveness of the adopted optimization.

The optimized degree distributions, for the AWGN and the phase noise channel, are substantially different and codes designed for the AWGN channel exhibit a significant loss on the phase noise channel. An example is shown in Table II where, for a phase noise channel with $\sigma_\Delta = 6$ degrees, the parameters of the optimized codes are shown for two values of E_S/N_0 and compared with those of the codes optimized for the AWGN channel. In this case also, the maximum variable node degree has been set to 25 and the minimum to 3. The corresponding code rates are also reported along with the optimized value of d_c.

The fact that codes designed for the AWGN and the phase noise channel are substantially different, is also mentioned in [14] and has been observed for the LDPC codes of the DVB-S2 system which were designed for the AWGN channel—they are highly suboptimal on a channel with strong phase noise, at least when the performance is phase noise limited [4].

VII. CONCLUSIONS

The evaluation of the information rate for a channel affected by a Wiener phase noise has been discussed. The derivation is based upon the approach in [13] for channels with memory. The results obtained by computer simulations are very interesting, since the presence of a phase noise incurs an energy loss which is in line with the loss observed in practical coded systems. Moreover, a simple closed-form bound for the information rate of PSK modulations has been presented. Finally, a procedure for the optimization of irregular repeat-accumulate codes was carried out, based on EXIT charts, showing that codes with negligible losses with respect to the capacity can be designed also for the channel with phase noise.

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