On Information Theoretic Aspects of Single- and Multi-Carrier Communications

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***Abstract*—In this paper, we investigate the relation between the quadrature amplitude modulation (QAM) input information rates (IRs) of multi-carrier (MC) and single carrier (SC) systems transmitted over inter-symbol interference (ISI) channels with additive white Gaussian noise (AWGN). In particular, considering uniform power spectrum transmission, we conjecture that, for a given channel impulse response, the IR corresponding to an SC input distribution is higher than that corresponding to an MC input distribution. We give an intuitive justification of our conjecture and confirm it, by means of numerical results, considering two sets of randomly generated channels.**

I. Introduction

Multi-carrier (MC) modulation for dispersive additive white Gaussian noise (AWGN) channels is known to achieve the channel capacity by means of water-filling power allocation and Gaussian input distribution [1]. Water-filling, nevertheless, needs channel knowledge at the transmitter, which has to be obtained through a feedback channel. In the absence of channel knowledge, a reasonable choice is to transmit a uniform power spectrum in the allocated system bandwidth. This situation may arise, for example, in the case of broadcast channels, or whenever power spectrum shaping techniques entail a prohibitive computational complexity.

The choice of the modulation technique (either single-carrier, SC, or MC), the particular constellation, as well as the possible spectral shaping technique, have an impact on the distribution at the input of the channel, and this affects, in return, the information rate (IR) of the system. In [2]–[5], a numerical method for computing upper and lower bounds and asymptotically accurate estimates of the IR for inter-symbol interference (ISI) channels with arbitrary Markov chain inputs is given. This method allows unprecedented accuracy in the computation of the IR of finite memory channels. In [6], [7], the authors propose upper and lower bounds for the IR of SC systems where independent and identically distributed (i.i.d.) symbols, chosen from an arbitrary constellation, are transmitted over ISI channels.

In this paper, we conjecture the existence of a novel lower bound for the IR of SC systems with uniform input power spectrum transmission over an ISI channel. In particular, we focus on quadrature amplitude modulation (QAM). We show, by means of numerical results, that, for fixed channel signal-to-noise ratio (SNR) and transmitted power with uniform spectral distribution, the IR of an SC scheme is higher than that of an MC scheme using the same QAM modulation on every subcarrier. This conclusion is not verified if water-filling and bit-loading techniques are considered in MC schemes.

The paper outline is as follows. In Section II, we describe the considered system model. In Section III, after briefly surveying previously derived results, we conjecture the novel inequality relating the IRs of SC and MC schemes. In Section IV, we present numerical results supporting our conjectured bound. In Section V, the performance improvement of MC schemes, obtained with water-filling power spectrum allocation and bit-loading, is investigated. In Section VI, some concluding remarks are drawn.

II. Preliminary Considerations and System Model

In this paper, we consider two systems classes transmitting i.i.d. QAM symbols over an ISI channel as shown in Fig. 1. Fig. 1 (a) refers to a SC system, where $\{h_k\}$ denotes the impulse response of the channel. The input symbols $\{\boldsymbol{x}_k\}$ are drawn i.i.d. from a QAM constellation, $\{\boldsymbol{w}_k\}$ is an i.i.d. sequence of zero-mean Gaussian samples, i.e., the noise samples and $\{\boldsymbol{y}_k\}$ is the channel output. Fig. 1 (b) refers to a MC system, the only difference with respect to the SC system being the presence of an inverse discrete Fourier transform (IDFT) block performing an (ideally infinite)[1] orthogonal frequency division multiplexing (OFDM) modulation. The output of the MC system is the sequence of samples $\{\boldsymbol{z}_k\}$.

In the case of a SC system, the output sample at epoch k can be expressed as follows:

$$\boldsymbol{y}_k = \sum_{l=0}^{L} h_l \boldsymbol{x}_{k-l} + \boldsymbol{w}_k \tag{1}$$

where $L+1$ is the number of consecutive nonzero samples in the channel impulse response. The channel output at epoch k in the case of a MC system can be written as

$$\boldsymbol{z}_k = \sum_{l=0}^{L} h_l \tilde{\boldsymbol{x}}_{k-l} + \boldsymbol{w}_k \tag{2}$$

where $\tilde{\boldsymbol{x}}_k$ denotes the sample output by the MC modulator at epoch k. In this paper, we focus on ideal MC modulation, i.e., each subchannel can be considered as an AWGN channel with an attenuation which depends on the subchannel index. This

[1] We assume that in each subcarrier the channel can be considered flat. This condition is satisfied in the limit for an infinite number of zero-bandwidth subcarriers.

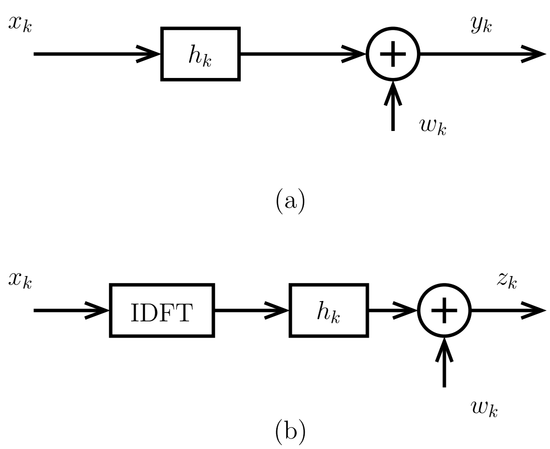

Fig. 1. Considered communication system models: (a) SC and (b) MC.

can be obtained using, for example, OFDM systems with a suitably large number of carriers.

The IR of a channel is given by the following expression [1], [6]:

$$I(\mathcal{X};\mathcal{Y}) = \lim_{n\to\infty} \frac{1}{n} I(\boldsymbol{y}^{(n)};\boldsymbol{x}^{(n)}) = h(\mathcal{Y}) - h(\mathcal{Y}|\mathcal{X}) \tag{3}$$

where $\boldsymbol{x}^{(n)} = (\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{n-1})$, $\boldsymbol{y}^{(n)} = (\boldsymbol{y}_0,\ldots,\boldsymbol{y}_{n-1})$, $h(\mathcal{Y})$ denotes the (differential) entropy rate of the channel output process $\mathcal{Y} = \{\boldsymbol{y}_k\}$, and $h(\mathcal{Y}|\mathcal{X})$ denotes the (differential) entropy rate of the channel output given the channel input. In particular, $h(\mathcal{Y}|\mathcal{X})$ is equal to the differential entropy of the noise process $\mathcal{W}$, namely, $h(\mathcal{W}) = \log 2\pi e\sigma^2$, where σ^2 denotes the per-dimension variance of the noise sample. The IR in (3), in the case of SC modulation, may be evaluated using Monte Carlo simulation techniques [2]–[5], since its analytical evaluation is, in general, a formidable task. We will refer to the IR of an SC scheme using the notation $I_{\mathbf{SC}}$.

In the case of a MC channel, (3) simplifies to the following integral expression, which assumes the use of an infinite number of carriers:

$$I_{\mathbf{MC}} = \frac{1}{2\pi}\int_0^{2\pi} I_{\mathrm{QAM}}[\gamma(\omega)]\,\mathrm{d}\omega \tag{4}$$

where $I_{\mathrm{QAM}}(\cdot)$ is the IR, as a function of the SNR, of a memoryless AWGN channel with the considered QAM input constellation and $\gamma(\omega)$ is the SNR at the receiver as a function of the frequency and is given by

$$\gamma(\omega) = \frac{E_{\mathrm{s}}}{N_0}|H(\omega)|^2 \tag{5}$$

where E_{s} is the average transmitted symbol energy, $N_0 = E\{|w_k|^2\}$ is the monolateral noise power spectrum and $H(\cdot)$ denotes the channel frequency response. In the following the system SNR will be defined as the ratio E_{s}/N_0.[2]

[2]We remark that the same SNR definition was used in [8], although it was erroneously stated that SNR was the ratio between the average *received* sample energy and N_0. The two definitions, i.e., referring to average received and transmitted sample enery, are equivalent in the case of normalized channel impulse response energy and uniform transmitted power spectrum.

Provided that each subchannel exhibits an almost flat frequency response, the IR of a MC channel can be approximately computed considering a finite number of channels as follows:

$$I_{\mathbf{MC}} \simeq \frac{1}{N}\sum_{i=0}^{N-1} I_{\mathrm{QAM}}[\gamma(2\pi i/N)]\,. \tag{6}$$

III. A Conjectured Lower Bound for the IR of SC Channels

The IR of a SC scheme, given by expression (3), has been thoroughly investigated in several works. In [7], the authors give a survey of existing lower and upper bounds and propose new lower bounds for $I_{\mathbf{SC}}$. In particular, both upper and lower bounds for $I_{\mathbf{SC}}$ in scenarios with ISI channels are given in terms of the IR of memoryless AWGN channels with the same input distribution:

$$\begin{aligned} I_{\mathbf{SC}} &\leq I(\boldsymbol{x}_k;\xi\boldsymbol{x}_k + \boldsymbol{w}_k) & (7)\\ I_{\mathbf{SC}} &\geq I(\boldsymbol{x}_k;\rho\boldsymbol{x}_k + \boldsymbol{w}_k) & (8)\end{aligned}$$

where

$$\xi \triangleq \max_{0\leq\lambda\leq\pi}|H(\lambda)|$$

for the upper bound and

$$\rho \triangleq \exp\frac{1}{2\pi}\int_0^{\pi}\ln|H(\lambda)|^2\,\mathrm{d}\lambda$$

for the lower bound.[3] It can be easily shown that MC and SC schemes yield the same IR in the case of memoryless AWGN channel [8]. As a consequence, the IRs $I(\boldsymbol{x}_k;\xi\boldsymbol{x}_k+\boldsymbol{w}_k)$ and $I(\boldsymbol{x}_k;\rho\boldsymbol{x}_k+\boldsymbol{w}_k)$ in (7) and (8), respectively, can also be interpreted as the IRs of two memoryless AWGN channels with MC input. Another bound provided in [7] is the so-called minimum mean square error (MMSE) lower bound, which states that

$$I_{\mathbf{SC}} \leq I(\boldsymbol{x}+\boldsymbol{\nu};\boldsymbol{x})$$

where $\boldsymbol{\nu}$ is a properly defined non-Gaussian noise term. In [7], a conjectured bound is also provided on the basis of the MMSE lower bound:

$$I_{\mathbf{SC}} \leq I(\boldsymbol{x}+\hat{\boldsymbol{\nu}};\boldsymbol{x})$$

where $\hat{\boldsymbol{\nu}}$ is a Gaussian random variable with $\mathrm{Var}\{\hat{\boldsymbol{\nu}}\} = \mathrm{Var}\{\boldsymbol{\nu}\}$.

Our goal is to derive a novel lower bound for $I_{\mathbf{SC}}$. In particular, we want to relate $I_{\mathbf{SC}}$ with the IR of a channel with the same i.i.d. input process transmitted by means of a MC modulation scheme. In other words, we want to relate $I_{\mathbf{SC}}$ and $I_{\mathbf{MC}}$.

Let us consider the received signal in the frequency domain. Assuming DFT-based demodulation with DFT block length equal to N,[4] the signal at the output of the demodulator, in

[3]Note that both bounds refer to an ISI channel with real taps.

[4]We remark that DFT isolates perfectly separated channels in the frequency domain only asymptotically, i.e., for a number of subcarriers that tends to infinity.

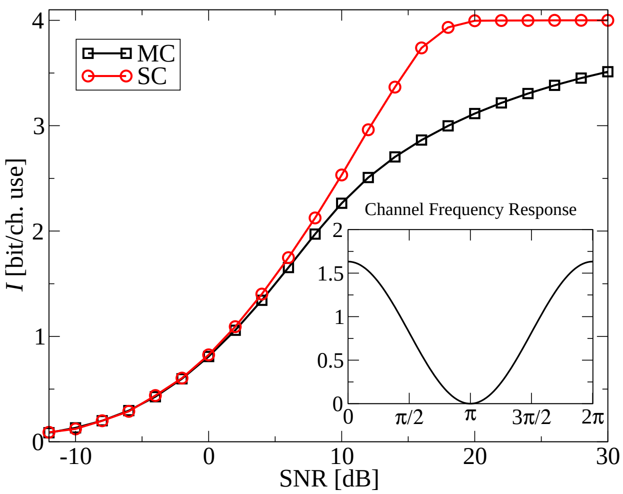

Fig. 2. $I_{\rm SC}$ and $I_{\rm MC}$, as functions of the SNR, considering an ISI channel with impulse response $(1,2,1)/\sqrt{(6)}$. The channel frequency response, which presents a spectral zero at frequency π, is also shown inside the figure.

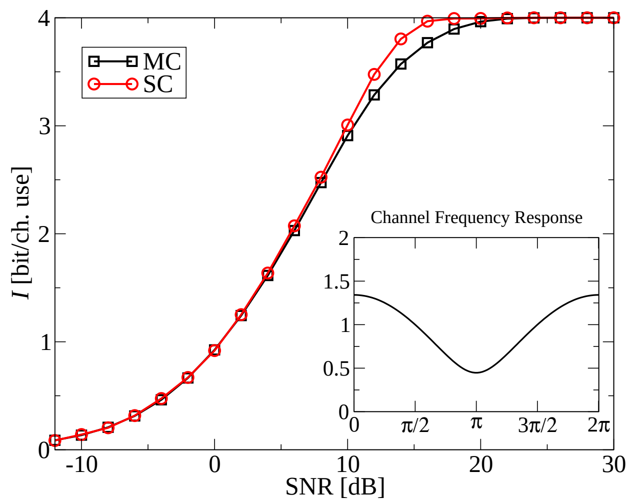

Fig. 3. $I_{\rm SC}$ and $I_{\rm MC}$, as functions of the SNR, considering a channel with impulse response $(2,1)/\sqrt{(5)}$. The channel frequency response is also shown inside the figure.

the case of SC modulation is given by

$$\begin{aligned}\tilde{y}_k &= \frac{1}{N}\sum_{i=0}^{N} y_i e^{-j2\pi ki/N} \\ &= \frac{1}{N}\sum_{i=0}^{N}\sum_{l=0}^{L} h_l x_{i-l} e^{-j2\pi ki/N} + \frac{1}{N}\sum_{i=0}^{N} w_i e^{-j2\pi ki/N} \\ &= \frac{1}{N}\sum_{i=0}^{N}\sum_{l=0}^{L} h_l x_{i-l} e^{-j2\pi ki/N} + \tilde{w}_k \end{aligned} \tag{9}$$

where $\{h_k\}$ is the channel impulse response and $\tilde{w}_k$ has the same distribution of w_k. If the channel impulse response length $L+1$ is much smaller than N, by periodicizing both $\{h_k\}$ and $\{x_k\}$ with period N, it is possible to accurately approximate (9) with the following:

$$\tilde{y}_k = \frac{1}{N} H_k \sum_{i=0}^{N} x_i e^{-j2\pi ki/N} + \tilde{w}_k \tag{10}$$

where H_k denotes the k-th element of the length-N DFT of the channel impulse response. This approximation impacts only on the samples at the border of the block, which should depend also on adjacent blocks [6]. We remark that, since a DFT is an invertible operation, the IR $I(\mathcal{Y};\mathcal{X})$ is equal to $I(\tilde{\mathcal{Y}};\mathcal{X})$. Therefore, in the following we will use (10) to draw come considerations on the IR of the SC scheme. In the case of MC modulation, the demodulator output becomes

$$\begin{aligned}\tilde{z}_k &= \frac{1}{N}\sum_{i=0}^{N} z_i e^{-j2\pi ki/N} \\ &= H_k x_k + \tilde{w}_k\,. \end{aligned} \tag{11}$$

The frequency domain representations (10) and (11) suggest the presence of a bottleneck for the IR of MC schemes. In fact, whereas each received SC sample $\tilde{y}_k$ depends on all transmitted symbols $\{x_0,\ldots,x_{N-1}\}$, the received MC sample $\tilde{z}_k$ depends only on the transmitted symbol x_k. Therefore, in the MC case the maximum contribution of a subcarrier to the total IR is limited by the logarithm of the constellation cardinality, regardless of the particular attenuation (or amplification) of the subchannel. This is not the case for SC systems, since each sample $\tilde{y}_k$ in (10) depends on every symbol in the sequence and may contribute to the overall IR up to the total number of transmitted bits. In this sense, an SC scheme transmits the entire message through every subchannel and the receiver may exploit this "frequency diversity" by performing optimum decision based on all subchannel outputs. In particular, subchannels with high SNR do not undergo the limitations due to the constellation cardinality suffered by the subchannels in MC schemes. We can then state our conjectured bound as follows:

$$I_{\rm SC} \geq I_{\rm MC} \tag{12}$$

where $I_{\rm SC}$ and $I_{\rm MC}$ have the expressions given by (3) and (4), respectively. Note that expression (4) holds for a QAM input distribution. However, our conjecture is that (12) holds for any input constellation.

IV. Numerical Evidence of the Conjectured Bound

In order to get insights into the relation between the IRs of MC and SC schemes, we preliminarily evaluate $I_{\mathbf{SC}}$ and $I_{\mathbf{MC}}$ considering ISI channels with impulse responses $(1,2,1)/\sqrt{(6)}$ and $(2,1)/\sqrt{(5)}$. The corresponding results are shown in Fig. 2 and 3, respectively. The frequency responses of the channels are also shown in the same figures. The $(1,2,1)/\sqrt{(6)}$ channel has a spectral zero at angular frequency equal to π, whereas the $(2,1)/\sqrt{(5)}$ channel has a frequency response strictly larger than zero. Clearly, in both scenarios the SC curve is higher than the MC curve.

The impact of the presence of a spectral zero on $I_{\rm MC}$ is further investigated in Fig. 4, where $I_{\rm MC}$ is evaluated for ISI channels with impulse responses $(1,1)/\sqrt{2}$, $(1,2,1)/\sqrt{6}$, $(1,3,3,1)/\sqrt{20}$ and $(1,4,6,4,1)/\sqrt{70}$ associated with a 1st, 2nd, 3rd, and 4th order spectral zero at angular frequency equal

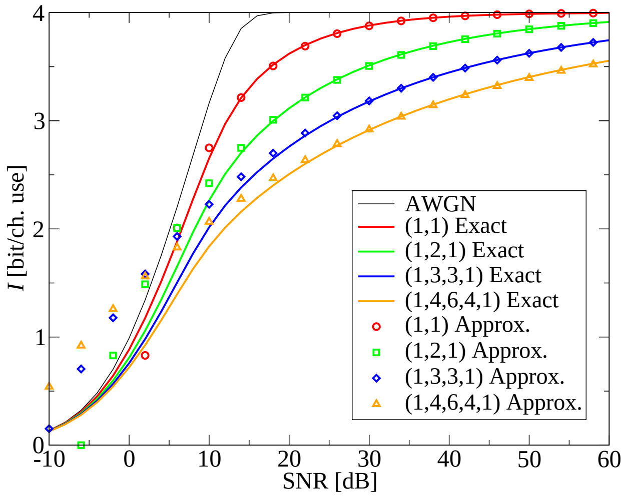

Fig. 4. I_{MC}, as a function of the SNR, considering several ISI channels with a (multiple) spectral zero at angular frequency equal to π.

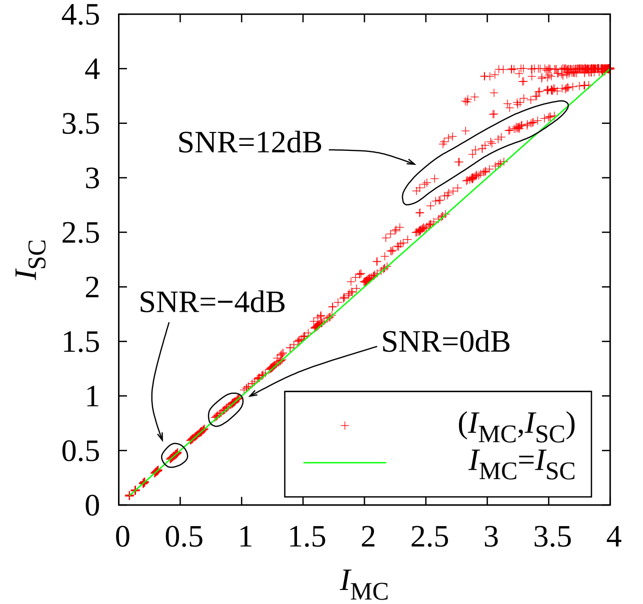

Fig. 5. I_{SC}, as a function of I_{MC}, for 40 randomly generated *real* ISI channels with 3-tap impulse responses. Various SNR values are also considered. Each cross corresponds to a $(I_{\mathrm{SC}}, I_{\mathrm{MC}})$ relative to a specific channel at a specific SNR. A few characteristic patterns, associated to given SNR values, are highlighted.

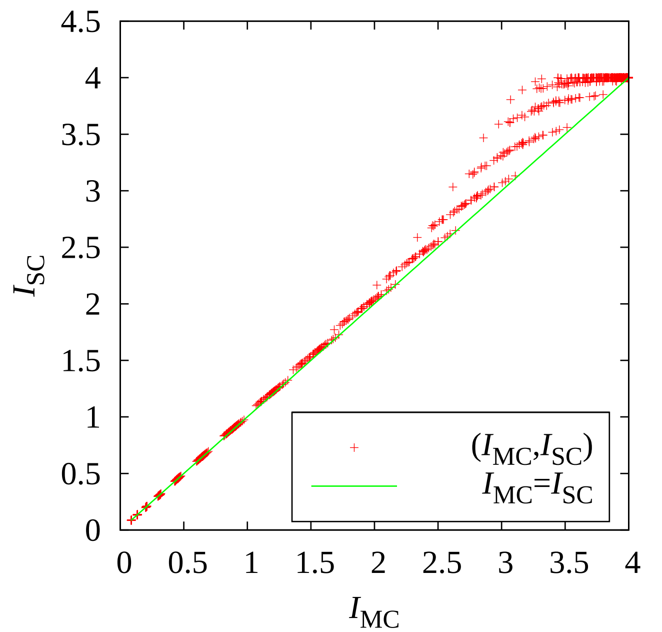

Fig. 6. I_{SC}, as a function of I_{MC}, for 40 randomly generated *complex* ISI channels with 3-tap impulse responses. Various SNR values are also considered. Each cross corresponds to a $(I_{\mathrm{SC}}, I_{\mathrm{MC}})$ relative to a specific channel at a specific SNR.

to π, respectively. In particular, I_{MC} approaches its maximum value with the following approximate law:

$$I_{\mathrm{MC}}(\mathrm{SNR}) \simeq I_{\max}\big(1 - \alpha_n \mathrm{SNR}^{-\frac{\beta}{n}}\big) \tag{13}$$

where n is the order of the spectral zero, α_n is a parameter which depends on n, β is a constant, and $I_{\max}$ is the logarithm of the cardinality of the considered constellation. As one can see from Fig. 4, the approximate expression (13) is very tight at large SNR values.

In order to verify our conjecture (12), we randomly generate 40 real ISI channels and 40 complex ISI channels, all with length-3 impulse responses. The channel taps are i.i.d. zero mean and unit variance Gaussian samples, either real or complex. Each channel impulse response is normalized to unit energy, in order to allow a fair comparison between different channels. In all cases, 16-QAM is the modulation format of choice. In the MC case, I_{MC} is computed by numerical integration of (4)—we point out, however, that the approximate computation based on (6), considering $N = 64$ or $N = 1024$ subchannels, entails a negligible difference. In the SC case, I_{SC} is computed using the algorithm in [5], and transmitting 10^6 QAM symbols. The considered SNR range is between -12 dB and 30 dB. In Fig. 5, I_{SC} is shown as a function of I_{MC}, for various real channels and SNR values. The $(I_{\mathrm{SC}}, I_{\mathrm{MC}})$ point corresponding to a specific channel-SNR pair is represented by a single cross in the plot. For comparison purposes, the curve $I_{\mathrm{SC}} = I_{\mathrm{MC}}$ is also shown. Clearly, the results show that, in the considered scenarios, $I_{\mathrm{SC}} \geq I_{\mathrm{MC}}$. In particular, the difference seems larger in the medium-high IR region, i.e., just before the saturation at the value $4 = \log_2 16$. The validity of the conjectured bound (12) is confirmed, in the presence of complex ISI channels, by the results in Fig. 6.

In Fig. 7, the difference $I_{\mathrm{SC}} - I_{\mathrm{MC}}$ is shown, for each real ISI channel, as a function of the SNR. As expected, the difference is always non-negative. Furthermore, there is a common behavior followed by every channel, summarized in the following remarks.

- At low SNR, the difference between the IRs of SC and MC schemes is negligible. This is due to the fact that in the low power regime every symmetric input provides the same performance.
- In the medium SNR region, where the transition from low spectral efficiency to almost maximum spectral efficiency is observed, the difference between I_{SC} and I_{MC} becomes largest. This is the region where the difference between SC and MC input distribution has its maximum impact.

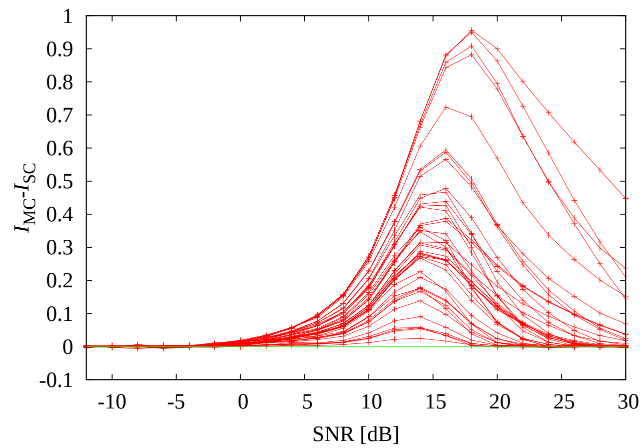

Fig. 7. $I_{\rm SC} - I_{\rm MC}$ versus SNR for several random real channels with 3-tap impulse response. Each curve is associated to a specific channel.

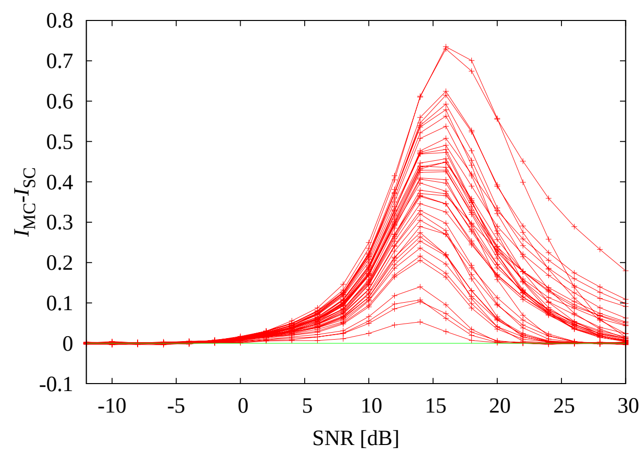

Fig. 8. $I_{\rm SC} - I_{\rm MC}$ versus SNR for several random complex channels with 3-tap impulse response. Each curve is associated to a specific channel.

- In the high SNR region, since the IRs of both SC and MC schemes saturate at the logarithm of the cardinality of the used constellations, the difference becomes minimal. However, depending on the channel, this saturation may occur at very high SNR values. In particular the channels characterized by deep notches or highly attenuated frequency ranges exhibit a slow convergence to the maximum IR value, in agreement with the observation, made at the beginning of this section, regarding the presence of spectral zeros.

In Fig. 8, the difference $I_{\rm SC} - I_{\rm MC}$ is shown, for each complex channel, as a function of the SNR. The results confirm those obtained for the real channels. In particular, for complex channels the difference between the IRs of SC and MC schemes is smaller compared to the case with real channels. This is due to the fact that, with the considered channel tap statistics, real channels may present, with higher probability with respect to complex ones, deep spectral attenuations or quasi-zeros. In general, channels characterized by strongly attenuated spectral regions exhibit a larger difference between $I_{\rm SC}$ and $I_{\rm MC}$.

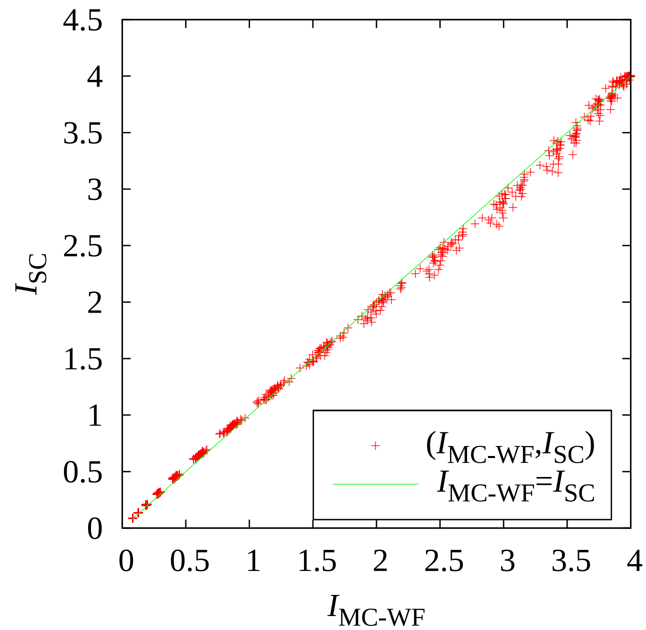

Fig. 9. $I_{\rm SC}$ versus $I_{\rm MC-WF}$ for several randomly generated ISI real channels with 3-tap impulse responses.

V. Power Spectrum Allocation and Bit-Loading

In order to reduce the difference between the IRs of MC and SC schemes, two strategies can be followed. The first strategy consists in increasing the cardinality of the constellation. This allows to effectively exploit all the information that the high SNR MC subchannels may convey. In practice, if none of the MC subchannels is characterized by an SNR sufficiently high to saturate its memoryless IR curve, the difference between the IRs of SC and MC schemes is minimal, as can be seen in the low SNR region. In particular, numerical analysis shows that, increasing the cardinality of the constellation, for a given SNR, closes the gap between $I_{\rm MC}$ and $I_{\rm SC}$.

A second strategy consists of applying a water-filling technique to obtain an optimal transmit spectral shaping [1]. Usually water-filling is applied together with a suitable algorithm for per-channel constellation cardinality allocation, also known as *bit loading*. Typically, bit-loading algorithms allocate in each channel the highest possible number of bits that guarantees a given operational bit error rate (BER). In order to make a fair comparison between uniform input power SC and MC schemes with water-filling, however, we use a bit-loading algorithm (first introduced in [8]) which optimally allocates a given fixed number of bits to be transmitted by maximizing the overall system IR.[5] This allows to fix the signal space cardinality, thus limiting the system spectral efficiency. In Fig. 9, $I_{\rm SC}$ is shown as a function of the $I_{\rm MC-WF}$ of the MC scheme with water-filling and the above mentioned bit-loading algorithm, considering the above described 40 channels with 3 complex valued taps impulse response. The number of subcarriers is 64 and the average number of bits per subcarrier is 4. For comparison purposes, the line $I_{\rm SC} = I_{\rm MC-WF}$ is also

[5]For simplicity, increments of 1 bit per dimension are considered, by restricting to square QAM constellations.

shown. The algorithm using bit-loading and water-filling has slightly better IR performance than the uniform-input power spectrum SC scheme. However, the performance increase is smaller than the average performance increase of SC schemes, with respect to MC schemes, shown in Fig. 5. In practice, SC schemes seem to achieve a close-to-optimum performance.

VI. Conclusions

In this paper, we have conjectured a lower bound on IR for SC schemes transmitting an i.i.d. input through an ISI AWGN channel. In particular, the bound states that SC schemes have larger IR than the corresponding MC schemes, with the same input constellation and uniform input power spectrum. An intuitive justification of the bound, on the basis of an implicit diversity effect used by SC schemes, has been given. The conjectured bound has been numerically investigated using 80 randomly generated ISI channels with Gaussian taps. Our conjecture suggests that the SC modulation should be preferred to MC modulation whenever a feedback channel is not available, the channel is dispersive and may exhibit spectral nulls (or *quasi*-nulls), and the channel impulse response enables feasible SC receivers. In the presence of a proper bit-loading, however, MC schemes completely recover the IR loss with respect to SC schemes. Alternatively, a MC scheme with uniform input power can be used if large coding rates are not of interest.

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