Public Key (asymmetric) Cryptography

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Corso di Sicurezza nelle reti di telecomunicazioni, a.a. 2008/2009

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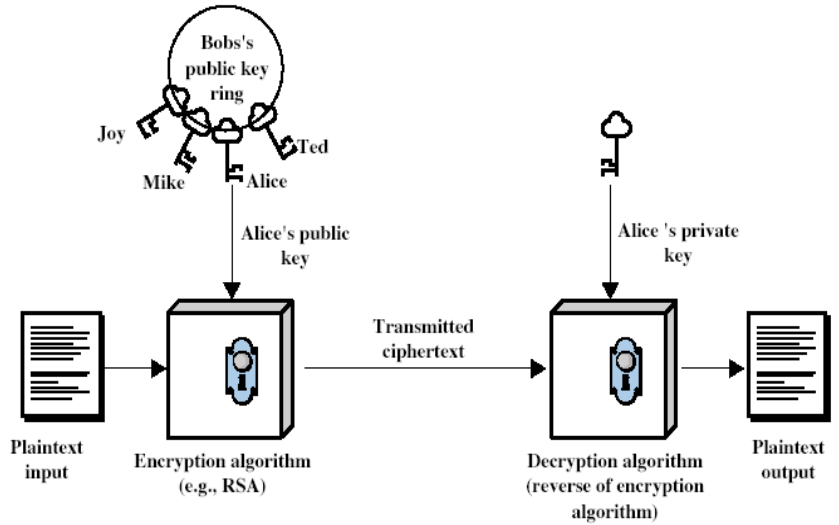
Public-Key Cryptography

- Sometimes also referred to as asymmetric cryptography (or two-key cryptography)
- Probably most significant advance in the 3000 year history of cryptography
- Public invention due to Whitfield Diffie & Martin Hellman in 1975
  - **at least that's the first published record**
  - **known earlier in classified community (e.g. NSA?)**
- Uses clever application of number theoretic concepts
- Complements rather than replaces private key crypto

Public-Key Cryptography

- With symmetric/secret-key cryptography
  - **you need a secure method of telling your partner the key**
  - **you need a separate key for everyone you might communicate with**
- Instead, with public-key cryptography, keys are not shared
- Public-key cryptography uses two keys:
  - **a public-key, which may be known by anybody, and can be used to encrypt messages, and verify signatures**
  - **a private-key, known only to the recipient, used to decrypt messages, and sign (create) signatures**
- Is asymmetric because
  - **those who encrypt messages or verify signatures cannot decrypt messages or create signatures**

Public-Key Cryptography



Public-Key Characteristics

- Public-Key algorithms rely on two keys with the characteristics that it is:
  - **computationally easy to en/decrypt messages when the relevant (en/decrypt) key is known**
  - **computationally infeasible to find decryption key knowing only algorithm & encryption key**
  - **either of the two related keys can be used for encryption, with the other used for decryption (in some schemes)**
- Public-Key algorithms rely on mathematic functions rather than on permutations and substitutions

Public-Key Characteristics

- Public key cryptography can do anything secret key cryptography can do, but..
- The known public-key cryptographic algorithms are orders of magnitude slower than the best known secret key cryptographic algorithms
  - **security of public-key algorithms still relies on key size (as for secret-key algorithms)**
  - **are usually only used for things secret key cryptography can't do (or can't do in a suitable way)**
- Often it is mixed with secret key technology
  - **e.g. public key cryptography might be used in the beginning of communication for authentication and to establish a temporary shared secret key used to encrypt the conversation**

Why Public-Key Cryptography?

- Can be used to:
  - key distribution **– secure communications without having to trust a KDC with your key (key exchange)**
  - digital signatures **–verify a message is come intact from the claimed sender (authentication)**
  - encryption/decryption **- secrecy of the communication (confidentiality)**
- Some algorithms are suitable for all uses, others are specific to one
- Note that public-key cryptography simplifies but not eliminates the problem of trusted systems and key management

Public-key Cryptography

- Note
  - **All secret key algorithms do the same thing: they take a block and encrypt it in a reversible way; there are chaining method to convert block ciphers into message ciphers**
  - **All the hash algorithms do the same thing: they take a message and perform an irreversible transformation on it**
  - **Instead, public key algorithms look very different**
    - in how they perform their function
    - in what functions they perform
- Example of public key algorithms:
  - **RSA, which does encryption and digital signature**
  - **El Gamal and DSS, which do digital signature but not encryption**
  - **Diffie-Hellman, which allows establishment of a shared secret**
  - **zero knowledge proof systems, which only do authentication**
- They all have in common: a private and a public quantities associated with a principal

Security of Public Key Schemes

- Like private key schemes brute force **exhaustive search** attack is always theoretically possible
- But keys used are too large (>512bits)
- A crucial feature is that the private key is difficult to determine from the public key
- Security relies on a **large enough** difference in difficulty between **easy** (en/decrypt) and **hard** (cryptanalyse) problems
- More generally the **hard** problem is known, its just made too hard to do in practise
  - **requires the use of** very large numbers
  - **hence is** slow **compared to private key schemes**

Rivest, Shamir, and Adleman (RSA)

Rivest, Shamir, and Adleman

- by Rivest, Shamir & Adleman of MIT in 1977
- best known & widely used public-key scheme
- Based on exponentiation in a finite (Galois) field over integers modulo n
  - **nb. exponentiation takes O((log n)^3) operations (easy)**
- uses large integers (eg. 1024 bits)
- security due to cost of factoring large numbers
  - **nb. factorization takes O(e $^{\log n \log \log n}$) operations (hard)**
- The key length is variable
  - **long keys for enhanced security, or a short keys for efficiency**
- The plaintext block size (the chunk to be encrypted) is also variable
  - **The plaintext block size must be smaller than the key length**
  - **The ciphertext block will be the length of the key**
- RSA is much slower to compute than popular secret key algorithms like DES and IDEA

Some Arithmetic

- **Relatively prime** - means that two values do not share any common factors other than 1
  - **eg. 8 & 15 are relatively prime since factors of 8 are 1,2,4,8 and of 15 are 1,3,5,15 and 1 is the only common factor**
- **totient function** - $\phi(n)$ - *(based on the words total and quotient)* The function tells how many numbers less than n are relatively prime to n; a.k.a. Euler's totient function
  - **to compute ø(n) need to count number of elements to be excluded**
    - e.g. $\phi(8) = |1,3,5,7| = 4$
  - **in general need prime factorization, but**
  - theo: **If n is prime, the all integers less than n (that is: 1, 2, ..., n-1) are relatively prime to n. Therefore $\phi(p) = p - 1$**
    - e.g. $\phi(37) = 36$
  - theo: **If n is the product of two primes (p and q) then there are (p-1)(q-1) numbers relatively prime to that quantity, that is $\phi(n) = \phi(p \cdot q) = (p-1)(q-1)$**
    - e.g. $\phi(21) = (3-1) \times (7-1) = 2 \times 6 = 12$

Some Arithmetic

- a mod $n=r_a$ | $a=q_a \cdot n + r_a$
- a=b mod n means that (a mod n) = (b mod n)
  - **i.e. a-b=k·n**
- properties:
  - **a op b mod n ≡ (a mod n) op (b mod n) = (a op b) mod n**
  - **with op = + , - , ***
- Some definitions in arithmetic modulo n
  - **complete set of residues is: 0..n-1**
  - **reduced set of residues is those numbers (residues) which are relatively prime to n**
    - **eg for n=10, complete set of residues is {0,1,2,3,4,5,6,7,8,9}**
    - **reduced set of residues is {1,3,7,9}**
  - **number of elements in reduced set of residues is the Euler Totient Function ø(n)**

Some Arithmetic

- **Multiplicative inverse** - the multiplicative inverse of a number x is the number we multiply x by to get 1. With real numbers this is just 1/x (for all but one case, guess which one that is) For our application we want to find:
- The multiplicative inverse of m mod n
  - **u | u*m = 1 mod n**
  - **this means what we want to find a number u so that u*m = 1 mod n**
  - **or we can say that u*m differs from 1 by a multiple of n, or u*m + v*n = 1**
  - **Euclid's algorithm can be used to solve this knotty problem. It only works if m and n are relatively prime**

Multiplicative Inverse Example

- Find the inverse of 797 mod 1047?
- That is, find u such that u * 797 = 1 mod 1047, or u*797 + v*1047 = 1
- Using Euclid's algorithm
  - **we get u = -490 and v = 373**
  - **That is: -490 * 797 + 373 * 1047 = 1**
  - **So (u*m) is -490 * 797 = -390530**
  - **The multiplicative inverse (u) is -490**
  - **-390530 mod 1047 = 1 mod 1047**

Fermat's Theorem

- $a^{p-1} \bmod p = 1$
  - **where p is prime, and**
  - **a is not divisible by p**
    - i.e. the greatest common divisor `gcd(a,p)=1`

Euler's Theorem

- A generalisation of Fermat's Theorem
- $a^{\phi(n)} \bmod n = 1$
  - **where gcd(a,n)=1**
- eg.
  - **a=3;n=10; ø(10)=4;**
  - **hence $3^4 = 81 = 1 \bmod 10$**
  - **a=2;n=11; ø(11)=10;**
  - **hence $2^{10} = 1024 = 1 \bmod 11$**

Euler's Theorem - Corollary

- a Corollary from Eulers's Theorem
- $a^{k\phi(n)+1} \bmod n = a$

RSA Algorithm

- First, you need to generate a public key and a corresponding private key. The scheme is:
  - Choose two large primes p and q (probably around 512 bits each).
  - Multiply them together (result is 1024 bits), and call the result n. The factors p and q will remain secret. (You won't tell anybody, and it's practically impossible to factor numbers that large.)
  - To generate your public key, choose a number e that is relatively prime (that is, it does not share any common factors other than 1) to $\phi(n)$. Since you know p and q, you know $\phi(n)$ -- it's (p-1)(q-1). Your public key is . <e, n>
  - To generate your private key, find the number d that is the multiplicative inverse of e mod $\phi(n)$. is your <d, n> private key.
  - To encrypt a message m (< n), someone using your public key should compute ciphertext . $c = m^e \bmod n$
  - Only you will be able to decrypt c, using your private key to compute $m = c^d \bmod n$

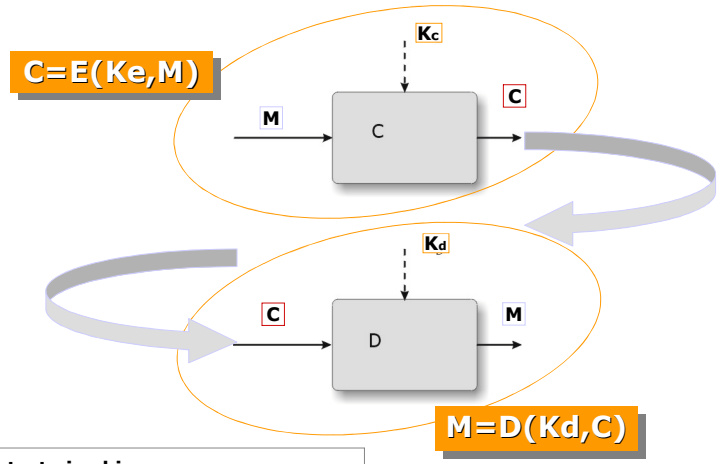
RSA Key Setup

- each user generates a public/private key pair by:
- selecting two large primes at random - p, q
- computing their system modulus N=p.q
  - **note ø(N)=(p-1)(q-1)**
- selecting at random the encryption key e
  - where 1<e<ø(N), gcd(e,ø(N))=1
- solve following equation to find decryption key d
  - **e.d=1 mod ø(N) and 0≤d≤N**
- publish their public encryption key: KU={e,N}
- keep secret private decryption key: KR={d,p,q}

RSA Use

- to encrypt a message M the sender:
  - **obtains** public key **of recipient** $KU=\{e,N\}$
  - **computes:** $C=M^e \bmod N$, **where** $0 \le M < N$
- to decrypt the ciphertext C the owner:
  - **uses their private key** $KR=\{d,p,q\}$
  - **computes:** $M=C^d \bmod N$
- note that the message M must be smaller than the modulus N (block if needed)

RSA



M Blocco di testo in chiaro
C Blocco di testo cifrato
Ke Chiave cifratura (e.g. chiave pubblica Ku)
Kd Chiave decifratura (e.g. chiave privata Kr)

Why RSA Works

- because of Euler's Theorem:
- $a^{\phi(n)} \bmod N = 1$
  - **where gcd(a,N)=1**
- in RSA have:
  - **N=p.q**
  - **ø(N)=(p-1)(q-1)**
  - **carefully chosen e & d to be inverses mod ø(N)**
  - **hence $e \cdot d = 1 + k \cdot \phi(N)$ for some k**
- hence :
  $C^d = (M^e)^d = M^{1+k\phi(N)} = M^1 \cdot (M^{\phi(N)})^q = M^1 \cdot (1)^q = M^1 = M \bmod N$

RSA Example

- Select primes: $p=17$ & $q=11$
- Compute $n = pq = 17 \times 11 = 187$
- Compute $\phi(n)=(p-1)(q-1)=16 \times 10 = 160$
- Select e : $\gcd(e,160)=1$; choose $e=7$
- Determine d: $de=1 \bmod 160$ and $d < 160$ Value is $d=23$ since $23 \times 7 = 161 = 10 \times 160 + 1$
- Publish public key $KU=\{7,187\}$
- Keep secret private key $KR=\{23,187\}=\{23,17,11\}$

RSA Example (cont)

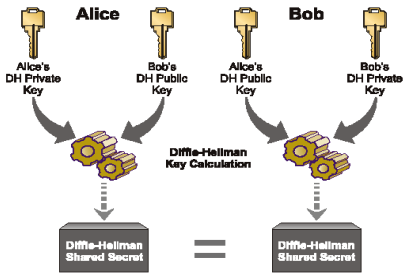
RSA encryption/decryption:

- given message `M = 88` (nb. `88<187`)
- encryption:
  $C = 88^7 \bmod 187 = 11$
- decryption:
  $M = 11^{23} \bmod 187 = 88$

RSA Security

- three approaches to attacking RSA:
  - **brute force key search (infeasible given size of numbers)**
  - **mathematical attacks (based on difficulty of computing ø(N), by factoring modulus N)**
  - **timing attacks (on running of decryption)**

Diffie-Hellman



Diffie-Hellman

- First public-key type scheme proposed
- by Diffie & Hellman in 1976 along with the exposition of public key concepts
  - **note: now know that James Ellis (UK CESG) secretly proposed the concept in 1970**
  - **predates RSA**
  - **Less general than RSA: it does neither encryption nor signature**
- is a practical method for public exchange of a secret key
  - **allows two individuals to agree on a shared secret (key)**
  - **It is actually used for key establishment**
- used in a number of commercial products

Diffie-Hellman Setup

- all users agree on global parameters:
  - **large prime integer or polynomial p**
  - **α g primitive root mod p**
- each user (eg. A) generates their key
  - **chooses a secret key (number): $x_A < p$**
  - **compute their** public key**: $y_A = g^{x_A} \bmod p$**
- each user makes public that key y_A

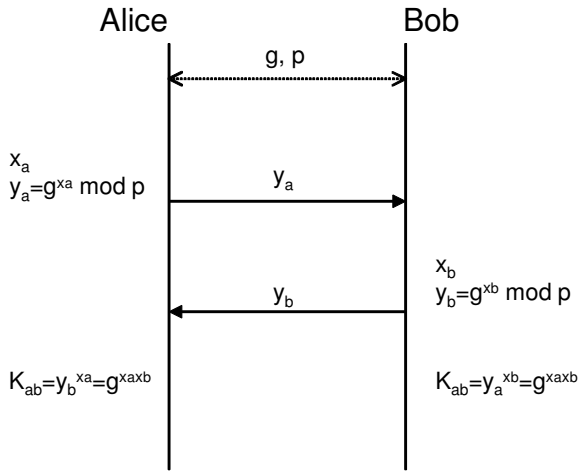
Diffie-Hellman Key Exchange

- shared session key for users A & B is K_{AB}:

$K_{AB} = g^{x_A \cdot x_B} \bmod p$
$= y_A^{x_B} \bmod p$ (which B can compute)
$= y_B^{x_A} \bmod p$ (which A can compute)

- K_{AB} is used as session key in private-key encryption scheme between Alice and Bob
- attacker must solve discrete log

Diffie-Hellman Key Exchange



Diffie-Hellman Example

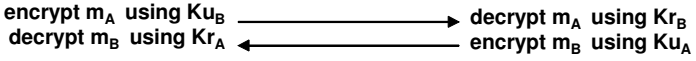
- users Alice & Bob who wish to swap keys:
- agree on prime p=353 and g=3
- select random secret keys:
  - **A chooses x_A=97, B chooses x_B=233**
- compute public keys:
  - **$y_A=3^{97} \bmod 353 = 40$ (Alice)**
  - **$y_B=3^{233} \bmod 353 = 248$ (Bob)**
- compute shared session key as:
  - **$K_{AB}= y_B^{x_A} \bmod 353 = 248^{97} = 160$ (Alice)**
  - **$K_{AB}= y_A^{x_B} \bmod 353 = 40^{233} = 160$ (Bob)**

Zero Knowledge Proof Systems

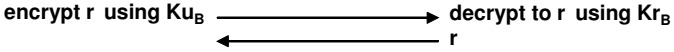
- Only do authentication
  - **prove that you know a secret without revealing the secret**
- RSA is a zero knowledge system
- There are zero knowledge systems with much higher performance
- Example (Isomorphic graphs):
  - **Alice defines two large (say 500 vertices) isomorphic graphs G_A , G_B**
  - **G_A and G_B become public, but only Alice knows the mapping**
  - **to prove her identity to Bob, Alice find a set of isomorphic graphs G_1,G_2, .. , G_k**
  - **Bob divides the set into two subset T_A and T_B**
  - **Alice shows to Bob the mapping between each $G_i \in T_A$ and G_A, and between each $G_j \in T_B$ and G_B**

Security uses of public key cryptography

- Transmitting over an insecure channel
  - **each party has a <public key, private key> pair (Ku,Kr)**
  - **each party encrypts with the public key of the other party**



- Secure storage on insecure media
  - **encrypt with public key, decrypt with private key**
  - **useful when you can let third party to encrypt data**
- Peer Authentication
  - **public key gives the real benefit**
  - **no n(n-1)/2 keys are needed**



Security uses of public key cryptography

- Data authentication (Digital signature)
  - **based on cryptographic checksum**
- Key establishment
  - **e.g. Diffie-Hellman**
- Note
  - **Public key cryptography has specific algorithm for specific function such as**
    - data encryption
    - MAC/digital signature
    - peer authentication
    - key establishment
- Remarks:
    - RSA encryption with the public key of the peer: encrypt
    - RSA encryption with the own private key: authenticate/sign
    - you can use both to ensure encryption and authentication/signature

Vantaggio dei sistemi a chiave pubblica

- Ogni utente deve mantenere solo un segreto (la propria chiave privata)
- Le chiavi pubbliche degli altri utenti possono essere mantenuti tramite infrastrutture intermediarie sicure (PKI)
- Il numero delle chiavi è proporzionale a N per la comunicazione reciproca tra N utenti

Digital signature and digital certification

Digital Signature

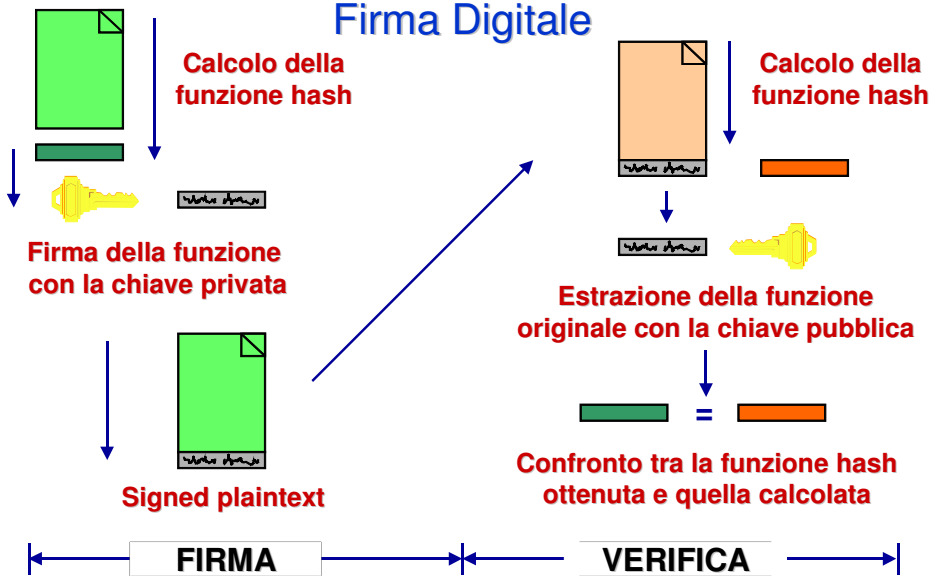
- Digital Signature is an application in which a signer, say "Alice," "signs" a message m in such a way that
  - **anyone can "verify" that the message was signed by no one other than Alice, and**
  - **consequently that the message has not been modified since she signed it**
- i.e. the message is a true and correct copy of the original
- The difference between digital signatures and conventional ones is that digital signatures can be mathematically verified
- The typical implementation of digital signature involves a message-digest algorithm and a public-key algorithm for encrypting the message digest (i.e., a message-digest encryption algorithm)

Digital Certification

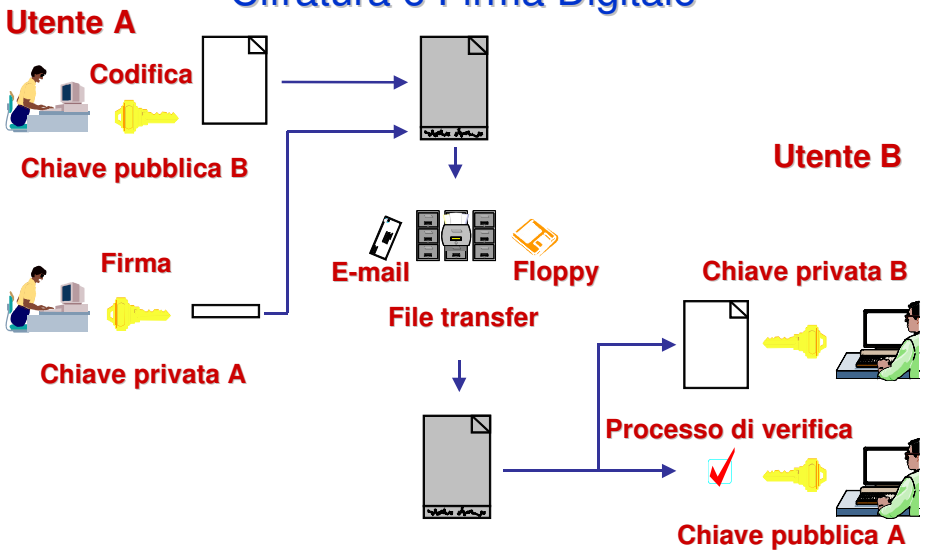
- Digital certification is an application in which a certification authority "signs" a special message m containing
  - **the name of some user, say "Alice," and**
  - **her public key**

  in such a way that anyone can "verify" that the message was signed by no one other than the certification authority and thereby develop trust in Alice's public key
- The typical implementation of digital certification involves a signature algorithm for signing the special message

Firma Digitale



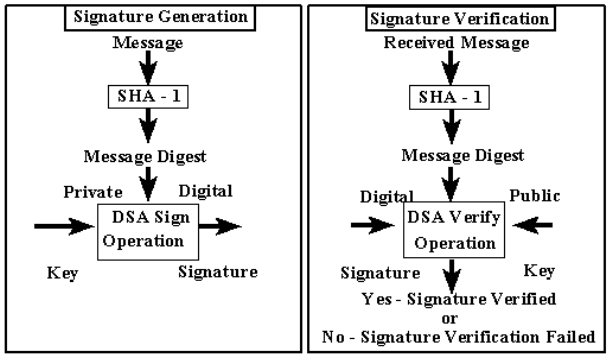
Cifratura e Firma Digitale



Digital Signature Standard (DSS)

- DSS (Digital Signature Standard)
- proposed by NIST (U.S. National Institute of Standards and Technology) & NSA in 1991
  - **FIPS 186**
- based on an algorithm known as DSA (Digital Signature Algorithm)
  - **DSS is the standard, DSA is the algorithm**
  - **the algorithm is a variant of the ElGamal scheme**
  - **uses 160-bit exponents**
  - **uses SHA/SHS hash algorithm**
  - **creates a 320 bit signature (160+160) but with 512-1024 bit security**
- security depends on difficulty of computing discrete logarithms

DSS Operations



DSA Key Generation

- have shared global public key values (p,q,g)
  - **L is the key length**
    - L = 1024 or more, and is a multiple of 64
  - **a large prime `p`**
  - **choose q, a 160 bit prime factor of p-1**
    - actually long as the hash H
  - **choose `g = h^((p-1)/q)`**
    - where `h<p-1, h^((p-1)/q) mod p > 1`
    - for some arbitrary h with 1<h<p-1
- choose `x<q`
- compute $y = g^x \bmod p$
- public key = (p,q,g,y)
- private key = x

DSA Signature Creation

- to sign a message `M` the sender generates:
  - **a random signature key `k, k<q`**
    - N.B.: `k` must be random, be destroyed after use, and never be reused
- computes the message digest:
  `h = SHA(M)`
- then computes signature pair:
  $r = (g^k \bmod p) \bmod q$
  $s = k^{-1}(h + x \cdot r) \bmod q$
- sends signature `(r,s)` with message `M`

DSA Signature Verification

- having received `M` & signature `(r,s)`
- to verify a signature, recipient computes:
  $w = s^{-1} \bmod q$
  $v = (g^{hw \bmod q}\, y^{rw \bmod q} \bmod p) \bmod q$
- if `v=r` then signature is verified
- proof
  $v = (g^{hw \bmod q}\, y^{rw \bmod q} \bmod p) \bmod q =$
  $= (g^{w(h+xr) \bmod q} \bmod p) \bmod q =$
  $= (g^k \bmod p) \bmod q =$
  $= r$