Statistics of the Jones matrix of fibers affected by polarization mode dispersion

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We carry out a statistical characterization of Jones matrix eigenvalues and eigenmodes to gain deeper insight into recently proposed fiber models based on Jones matrix spectral decomposition. A set of linear dynamic equations for the Pauli coordinates of the Jones matrix is established. Using stochastic calculus, we determine the joint distribution of the retardation angle of the eigenmodes and, indirectly, their autocorrelation function. The correlation bandwidth of the eigenmodes is found to be $\sqrt{2/3}$ that of the polarization mode dispersion vector. The results agree well with simulations performed with the standard retarded plate model. © 2001 Optical Society of America

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Optical fibers affected by polarization mode dispersion (PMD) are traditionally described in terms of PMD vector $\mathbf{\Omega}(\omega)$, whose statistics[1] and autocorrelation function[2,3] are known. Recent work on higher-order PMD[4] employed a model of the fiber Jones matrix that was based on its spectral decomposition, i.e., its eigenmodes and retardation angle, the statistical properties of which, to our knowledge, have never been investigated. Moreover, the same fiber model was recently used for PMD compensation.[5] Statistical characterization of the fiber eigenvalues and eigenmodes, as well as determination of their coherence bandwidth, is then helpful for comprehension of the evolution, in both frequency and length, of the fiber-system matrix and is the focus of this Letter. Dynamic equations for the fiber Pauli coordinates are first established. Such equations are then studied by use of the stochastic model of the local birefringence vector that was adopted in the research reported in Ref. 6.

A linear fiber of length z affected by PMD can be regarded as a two-input two-output linear system whose transfer matrix can, in the absence of polarization-dependent loss or gain, be written as the product of a scalar function and a unitary matrix $\boldsymbol{U}(z, \omega)$ with $\det[\boldsymbol{U}] = 1$. Matrix $\boldsymbol{U}$ can be written in the insightful form of a matrix exponential that was used in Ref. 7:

$$\begin{aligned}\boldsymbol{U}(z,\omega) &= \exp\left\{-i\,\frac{\Delta\phi(z,\omega)}{2}\,[\hat{\mathbf{b}}(z,\omega)\odot\boldsymbol{\sigma}]\right\}\\ &= u_0(z,\omega)\boldsymbol{\sigma_0} - i\check{\mathbf{u}}(z,\omega)\odot\boldsymbol{\sigma}\,,\end{aligned} \tag{1}$$

where i is an imaginary unit, $\odot$ stands for a scalar product, $\boldsymbol{\sigma_0}$ is the 2×2 identity matrix, $\boldsymbol{\sigma}$ is a formal vector whose components are the three Pauli spin matrices,[8] and we define $u_0(z, \omega) \triangleq \cos[\Delta\phi(z, \omega)/2]$ and $\check{\mathbf{u}}(z, \omega) \triangleq \sin[\Delta\phi(z, \omega)/2]\hat{\mathbf{b}}(z, \omega)$, where the real, unit-magnitude vectors $\pm\hat{\mathbf{b}}$ are the Stokes representations of the orthogonal eigenvectors of $\boldsymbol{U}$ associated with eigenvalues $\exp(\mp i\Delta\phi/2)$, where $\Delta\phi$ is the retardation angle.

Given the local birefringence Stokes vector $\mathbf{W}(z, \omega)$, we find that $\boldsymbol{U}' = (-i/2)[\mathbf{W}\odot\boldsymbol{\sigma}]\boldsymbol{U}$, where the prime indicates a derivative with respect to z. Hence the Pauli coordinates $[u_0', -i\check{\mathbf{u}}']$ of $\boldsymbol{U}'$ are derived from the Pauli coordinates $[u_0, -i\check{\mathbf{u}}]$ of $\boldsymbol{U}$:

$$\begin{aligned}\frac{\partial}{\partial z}u_0(z,\omega) &= -\frac{1}{2}\,\check{\mathbf{u}}(z,\omega)\odot\mathbf{W}(z,\omega)\,,\\ \frac{\partial}{\partial z}\check{\mathbf{u}}(z,\omega) &= -\frac{1}{2}\,\check{\mathbf{u}}(z,\omega)\times\mathbf{W}(z,\omega)\\ &\quad+\frac{1}{2}\,u_0(z,\omega)\mathbf{W}(z,\omega)\,,\end{aligned} \tag{2}$$

where $\times$ stands for vector cross product. These are the dynamic equations of the Pauli coordinates of $\boldsymbol{U}$.

As in Ref. 6, for fibers much longer than the correlation length, we assume that $\mathbf{W}(z, \omega) = \omega\sigma\mathbf{n}(z)$, where $\mathbf{n}(z)$ is a standard three-dimensional white-noise process. After translating Eqs. (2) into Ito form, we rewrite them in their canonical differential form[9] as $\mathrm{d}\mathbf{u} = \mathbf{c}(\mathbf{u})\mathrm{d}z + \boldsymbol{v}(\mathbf{u})\mathrm{d}\mathbf{B}(z)$, where $\mathbf{u} \triangleq [u_0, \check{\mathbf{u}}^T]^T$, $\mathrm{d}\mathbf{B}(z)$ is the differential of standard Brownian motion and the arrays

$$\begin{aligned}\mathbf{c}(\mathbf{u}) &\triangleq -\frac{3\omega^2\sigma^2}{8}\begin{pmatrix}u_0\\ \check{\mathbf{u}}\end{pmatrix},\\ \boldsymbol{v}(\mathbf{u}) &\triangleq \frac{\omega\sigma}{2}\begin{pmatrix}-\check{\mathbf{u}}^T\\ u_0\boldsymbol{\sigma_0}-[\check{\mathbf{u}}\times]\end{pmatrix},\end{aligned} \tag{3}$$

are the drift coefficient and the diffusion coefficient,[9] respectively. The canonical form is useful for applying the powerful tools of stochastic calculus.

Since $|\mathbf{u}|^2 = 1$, from the Fokker–Planck equation,[9] we find that the real vector $\mathbf{u}(z, \omega)$ has, for every z, a uniform first-order distribution on the unit sphere in $\mathcal{R}^4$. Hence, the spherical coordinates of $\mathbf{u}$ are found to be independent random variables with marginal densities:

$$p(\Delta\phi) = \frac{1-\cos(\Delta\phi)}{2\pi}\,,\qquad p(\theta) = \frac{1}{\pi}\,,\qquad p(\epsilon) = \cos(2\epsilon)\,, \tag{4}$$

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where $0 \leq \Delta\phi \leq 2\pi$ and the eigenmode $\hat{\mathbf{b}} = [\cos(2\theta)\cos(2\epsilon), \sin(2\theta)\cos(2\epsilon), \sin(2\epsilon)]^T$ has azimuth $-\pi/2 \leq \theta \leq \pi/2$ and ellipticity $-\pi/4 \leq \epsilon \leq \pi/4$. Thus, at fixed z and ω, $\hat{\mathbf{b}}(z, \omega)$ is independent of $\Delta\phi(z, \omega)$ and is uniformly distributed over the Poincaré sphere, as stated in Ref. 10 without proof. Moreover, the coordinates u_k, $k = 0, \ldots, 3$, have identical marginal densities $p(u_k) = (2/\pi) \times \sqrt{1 - u_k{}^2}(-1 \leq u_k \leq 1)$, and one can see that $E[u_k(z, \omega)] = 0$, for every $z > 0$, which implies that $E[\boldsymbol{U}] = \boldsymbol{0}$.

By augmenting Eqs. (2) with the joint vector $[\mathbf{u}(z, \omega_1), \mathbf{u}(z, \omega_2)]$ and applying the Martingale differential equation (also known as Dynkin's formula[9]), we also find that the coordinates u_k satisfy

$$E[u_j(z, \omega_1)u_k(z, \omega_2)] = \frac{1}{4}\exp\left(-\frac{3\sigma^2\Delta\omega^2 z}{8}\right)\delta_{jk}\,, \quad (5)$$

where $\Delta\omega = \omega_2 - \omega_1$ is the frequency deviation, δ_{jk} is the Kronecker delta function, and $j, k \in \{0, 1, 2, 3\}$. In deriving this result, we used the facts that $\mathbf{u}(z, \omega)$ is an asymptotically stationary stochastic process with respect to frequency ω and Eq. (5) is the asymptotic value. Strict sense stationarity does not hold, since in our model at $\omega = 0$ the birefringence vector is $\mathbf{W}(z, 0) = \mathbf{0}$ for every z along the fiber. From such a result we also get the autocorrelation function (ACF) of $\boldsymbol{U}(\omega)$ as follows. The Pauli coordinates of $\boldsymbol{U}^\dagger(\omega_1)\boldsymbol{U}(\omega_2)$ are $\{u_0(\omega_1)u_0(\omega_2) + \check{\mathbf{u}}(\omega_1) \odot \check{\mathbf{u}}(\omega_2), -i[u_0(\omega_1)\check{\mathbf{u}}(\omega_2) - u_0(\omega_2)\check{\mathbf{u}}(\omega_1) - \check{\mathbf{u}}(\omega_1) \times \check{\mathbf{u}}(\omega_2)]\}$, where † is a transpose conjugate. Using Eq. (5), one can see that only the zeroth component has a nonzero mean, so we get

$$E[\boldsymbol{U}^\dagger(\omega_1)\boldsymbol{U}(\omega_2)] = \exp\left(-\frac{3\sigma^2\Delta\omega^2 z}{8}\right)\boldsymbol{\sigma_0}\,, \quad (6)$$

extending to the Jones matrix a result already known for the Müller matrix.[2,3]

Since in Eqs. (2) optical frequency ω always appears multiplied by the standard deviation, σ, all the dynamic properties of Pauli coordinates in the frequency domain are expected to scale with σ, as confirmed by Eq. (5). Such σ is related to the mean-square magnitude of the fiber differntial group delay (DGD) by $\langle\Delta\tau^2\rangle = 3\sigma^2 z$. We can thus express the ACF of $\mathbf{u}$, using the mean-square DGD, as

$$R_{\mathbf{u}}(\omega_1, \omega_2) \triangleq E[\mathbf{u}^\dagger(z, \omega_1)\mathbf{u}(z, \omega_2)] = \exp\left(-\frac{\langle\Delta\tau^2\rangle\Delta\omega^2}{8}\right). \quad (7)$$

Finally, from Eqs. (3) and the theory of Brownian motion on spheres,[9] we are able to prove that the real vector $\mathbf{u}(z, \omega)$ describes in z a Brownian motion on the unit sphere in $\mathcal{R}^4$; the motion evolves at different rates for different values of optical frequency ω.

For the frequency derivative $\boldsymbol{U}_\omega$ of the Jones matrix, we find that $\boldsymbol{U}_\omega = (-i/2)[\boldsymbol{\Omega} \odot \boldsymbol{\sigma}]\boldsymbol{U}$. Thus, the same system of equations (2) holds if $\mathbf{u}'$ is replaced with $\mathbf{u}_{\boldsymbol{\omega}}$ and $\mathbf{W}$ with $\boldsymbol{\Omega}$, which shows that, given $\mathbf{u}$, $\mathbf{u}_{\boldsymbol{\omega}}$ is a linear combination of the components of $\boldsymbol{\Omega}$. Using Eqs. (2) and Poole's dynamic equation for the PMD vector, $\boldsymbol{\Omega}$, from the Fokker–Planck equation applied to the joint vector $[\mathbf{u}, \boldsymbol{\Omega}]$ we find that $\mathbf{u}$ and $\boldsymbol{\Omega}$ are independent random vectors for a given value of (z, ω). Hence from Ref. 1 we conclude that, given $\mathbf{u}$, $\mathbf{u}_{\boldsymbol{\omega}}$ is a Gaussian vector lying on the hyperplane orthogonal to $\mathbf{u}$, whereas for the marginal distribution of $\mathbf{u}_{\boldsymbol{\omega}}$ we have a Maxwellian magnitude and a uniform orientation in $\mathcal{R}^4$.

We simulated a set of 10,000 fibers to compare numerical results with theory. The simulations employed the standard retarded plate model, with each fiber realization consisting of $N = 100$ polarization-maintaining fiber plates. The plate's eigenmodes were linearly polarized, with uniform distribution on the Poincaré sphere equator. The retardation angle of each nth plate was $\Delta\phi_n(\omega) = \Delta\phi_{n0} + \Delta\phi_{n1}\omega$, with a random $\Delta\phi_{n0}$, uniform on $[0, 2\pi]$, and a fixed local DGD, $\Delta\phi_{n1} = \delta\tau$, chosen to give a rms DGD $\sqrt{\langle\Delta\tau^2\rangle} = \sqrt{N}\,\delta\tau$. In the simulations the number of plates must be large enough to avoid the frequency peridiocity of $\boldsymbol{U}(z, \omega)$ related to the fixed local DGD $\delta\tau$.

In Fig. 1 the theoretical probability density functions (4) of the retardation angle, the eigenmode azimuth, and the ellipticity are plotted and compared with simulation results, which represent averages over the fiber realizations at the reference frequency $\omega = 0$, and good agreement is found.

The ACF of vector $\mathbf{u}$ is plotted in Fig. 2 versus the normalized frequency difference $\Delta\omega\sqrt{\langle\Delta\tau^2\rangle}$,[11] along with that of the individual components u_k. Here

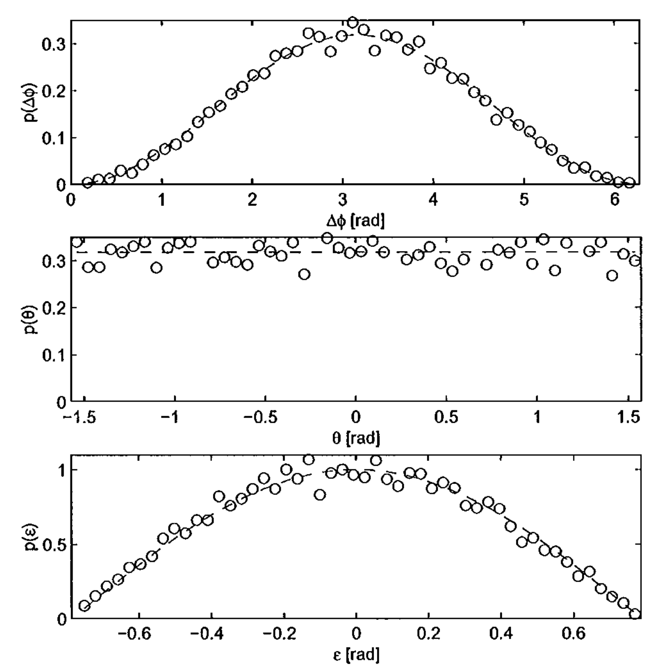

Fig. 1. Probability density of (top) retardation angle $\Delta\phi$, (middle) eigenmode azimuth θ, (bottom) eigenmode ellipticity ϵ. Dashed curves and line, theory [Eq. (4)]; circles, simulations.

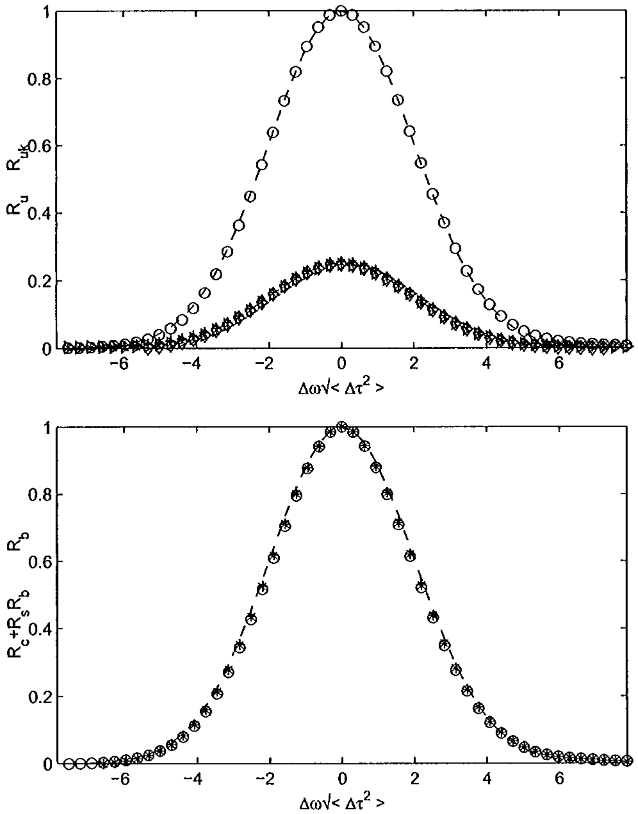

Fig. 2. top, ACF of (circles) vector **u** and other shapes its components, u_k; bottom, ACF of (asterisks) the eigenmode vector $\hat{\mathbf{b}}$ and (open circles) function $R_c + R_s R_{\hat{\mathbf{b}}}$. Dashed curves, theory.

again we note the good match of the simulations with theoretically predicted Gaussian functions (5) and (7). Comparison with the results in Refs. 2 and 3 shows that the correlation bandwidth of vector $\mathbf{u}(\omega)$ is $\sqrt{2/3} \cong 0.8$ times that of the PMD vector $\boldsymbol{\Omega}(\omega)$. Note that the ACF gives an indication of how quickly on average these vectors move in frequency with respect to their central frequency value but does not give information on the shape of the trajectories that the vectors are likely to follow.

Information on the ACF of retardation angle $\Delta\phi$ and eigenmode $\hat{\mathbf{b}}$ is buried in the ACF of **u**, and we did not find a way to extract it analytically. However, as was pointed out in Ref. 11, simulation with enough samples is in practice as good as theory determining ACF functions.

In Fig. 2 we again plot the theoretical ACF,

$$\begin{aligned} R_{\mathbf{u}}(\omega_1, \omega_2) \triangleq E\{&\cos[\Delta\phi(z, \omega_1)/2]\cos[\Delta\phi(z, \omega_2)/2] \\ &+ \sin[\Delta\phi(z, \omega_1)/2]\sin[\Delta\phi(z, \omega_2)/2] \\ &\times \hat{\mathbf{b}}(z, \omega_1) \odot \hat{\mathbf{b}}(z, \omega_2)\}, \end{aligned}$$

and function $R_c(\omega_1, \omega_2) + R_s(\omega_1, \omega_2)\, R_{\hat{\mathbf{b}}}(\omega_1, \omega_2)$, where the three terms are the simulated ACFs of $\cos[\Delta\phi(z, \omega)/2]$, $\sin[\Delta\phi(z, \omega)/2]$, and $\hat{b}(z, \omega)$. In Fig. 2 we also show the simulated ACF $R_{\hat{\mathbf{b}}}$ alone. From the figure we conclude that (i) $R_{\hat{\mathbf{b}}}(\omega_1, \omega_2) \cong R_{\mathbf{u}}(\omega_1, \omega_2)$. Hence, the correlation bandwidth of eigenmode $\hat{\mathbf{b}}$ is $\sqrt{2/3}$ that of the PMD vector $\boldsymbol{\Omega}$. Also, (ii) $\hat{\mathbf{b}}(z, \omega_1)$ and $\sin[\Delta\phi(z, \omega_2)/2]$ are independent for $\omega_1 = \omega_2$ and practically uncorrelated for any other choice of ω_1, ω_2.

We stress again that simulation and theory are in reasonable agreement, notwithstanding the remarkable differences between the retarded plate model used for the simulations and the theoretical model, in which the local birefringence direction has a uniform distribution on the Poincaré sphere, the local birefringence strength has a Maxwellian distribution, and there is no retardation at the reference frequency.

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