Tight bounds and accurate approximations for DQPSK transmission bit error rate

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Recently proposed bounds for the Marcum Q-function are applied to derive novel tight bounds rate for the bit error rate for differential quaternary phase shift keying transmission with Gray coding over an additive white Gaussian noise channel. Simple and accurate approximations are then proposed for the BER. The computational complexity of the final approximate expression is significantly lower than that of the exact expression.

Introduction: The focus of this Letter is the performance analysis of differential quaternary phase shift keying (DQPSK) transmission with Gray coding over the additive white Gaussian noise (AWGN) channel. This modulation format, due to its robustness to phase uncertainties, is applied in many practical communication systems [1]. The bit error rate (BER) in the case of Gray coding can be written in terms of the Marcum Q-function and the zeroth order modified Bessel function of the first kind, $I_0(x)$. In turn, the Marcum Q-function can also be expressed as an integral involving $I_0(x)$, which is itself defined by an integral expression. In this Letter, we show that it is possible to find very tight bounds and accurate approximate BER expressions, which completely avoid the computation of the Marcum Q-function. In particular, the final derived approximate expression entails only the computation of the exponential function. This derivation is based on the use of recently proposed bounds on the Marcum Q-function [2].

Preliminaries on Marcum Q-function: The Marcum Q-function is widely used in the performance analysis of digital communication systems [1]. The definition of this function is:

$$Q(a, b) \triangleq \int_b^\infty x \exp\left(-\frac{x^2 + a^2}{2}\right) I_0(ax)\, dx \tag{1}$$

It is known that the computation of the integral in (1) may lead to numerical problems, due to the semi-infinite integration region. For this reason, in [3, 4] the authors proposed alternative integral expressions for the Marcum Q-function with integration over a finite interval. In both cases, they consider suitable integral expressions for $I_0(x)$.

Instead of transforming the integration region of the integral in (1), upper and lower bounds for $I_0(x)$, extremely tight over the integration region, were introduced in [2]. Based on these bounds, tight bounds on the Marcum Q-function are derived for both cases with $b \geq a$ and $b < a$, respectively. As will be clear in the remainder of this Letter, the case with $b \geq a$ is of interest for the analysis of the BER expression of DQPSK with Gray coding. In this case, the bounds proposed in [2] are reported here for convenience:

$$Q(a, b) \leq \frac{I_0(ab)}{\exp(ab)} \left\{ \exp\left[-\frac{(b-a)^2}{2}\right] + a\sqrt{\frac{\pi}{2}} \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \right\} \tag{2}$$

$$Q(a, b) \geq \sqrt{\frac{\pi}{2}} \frac{I_0(ab) b}{\exp(ab)} \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \tag{3}$$

where $\operatorname{erfc}(x) \triangleq (2/\sqrt{\pi}) \int_x^\infty \exp(-t^2)\, dt$.

Novel bounds for BER of DQPSK with Gray coding: For DQPSK transmission with Gray coding over an AWGN channel, the BER can be expressed as:

$$\mathrm{BER} = Q(a, b) - \frac{1}{2} I_0(ab) \exp\left(-\frac{a^2 + b^2}{2}\right) \tag{4}$$

where $a = \sqrt{(2\gamma_b(1 - \sqrt{(1/2)}))}$, $b = \sqrt{(2\gamma_b(1 + \sqrt{1/2}))}$, and γ_b is the bit signal-to-noise ratio (SNR). Because $b > a$, the bounds introduced in (2) and (3) can be applied to derive the following bounds for the BER:

$$\mathrm{BER} \leq I_0(ab)\left[\sqrt{\frac{\pi}{2}} \frac{a}{\exp(ab)} \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) + \frac{1}{2}\exp\left(-\frac{a^2+b^2}{2}\right)\right] \tag{5}$$

$$\mathrm{BER} \geq I_0(ab)\left[\sqrt{\frac{\pi}{2}} \frac{a}{\exp(ab)} \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) + \frac{1}{2}\exp\left(-\frac{a^2+b^2}{2}\right)\right] \tag{6}$$

The novel BER bounds are shown in Fig. 1, together with the exact BER, as functions of the SNR. It can be immediately seen that the proposed bounds are extremely tight. Moreover, one also notes the striking formal symmetry between the upper bound (5) and the lower bound (6).

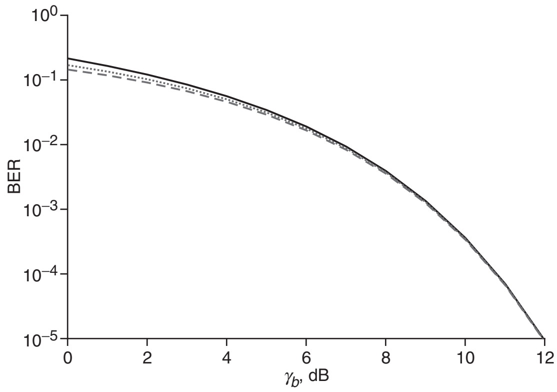

Fig. 1 *Upper and lower bounds for BER of DQPSK transmission with Gray coding over AWGN channel*

——— upper bound
.... exact
- - - - - - - lower bound

Approximate BER expressions for DQPSK with Gray coding: Based on the tight bounds (5) and (6) and because of their symmetry, it is intuitive to derive an approximate expression for the BER by simply considering the arithmetic average between these two bounds. This leads to the following expression:

$$\mathrm{BER}_{approx} \sqrt{\frac{\pi}{8}} \frac{I_0(ab)}{\exp(ab)} (a+b) \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \tag{7}$$

To estimate the tightness of the proposed approximate expression (7), it is expedient to define the error $\Delta \triangleq (\mathrm{BER}_{approx} - \mathrm{BER})$ and the percentage relative error $\varepsilon \triangleq 100 \times (\Delta/\mathrm{BER})$. Table 1 shows the exact BER, the approximate expression (7), Δ and ε for increasing values of the SNR. It can be immediately seen that the numerical difference between (4) and (7) is basically negligible ($\varepsilon < 1\%$) for all BER values of interest (say, $\mathrm{BER} < 10^{-2}$).

Table 1: Comparison, for increasing SNR, of exact BER, approximate BER expression (7), their difference and relative percentage error for DQPSK transmission over AWGN channel

γ_b, dB	BER	BER_{approx}	Δ	ε
0	0.16	0.17	0.01	6.2%
1	0.13	0.14	6.3×10^{-2}	4.8%
2	9.9×10^{-2}	0.10	3.4×10^{-3}	3.4%
3	7.2×10^{-2}	7.3×10^{-2}	1.6×10^{-3}	2.3%
4	4.8×10^{-2}	4.9×10^{-2}	7.2×10^{-4}	1.5%
5	3.04×10^{-2}	3.07×10^{-2}	2.9×10^{-4}	0.95%
6	1.72×10^{-2}	1.73×10^{-2}	1.0×10^{-4}	0.62%
7	8.5×10^{-3}	8.6×10^{-3}	3.5×10^{-5}	0.40%
8	3.64×10^{-3}	3.6×10^{-3}	9.8×10^{-6}	0.27%
9	1.267×10^{-3}	1.269×10^{-3}	2.3×10^{-6}	0.18%
10	3.432×10^{-4}	3.436×10^{-4}	4.0×10^{-7}	0.12%
11	6.790×10^{-5}	6.795×10^{-5}	5.2×10^{-8}	0.077%
12	9.053×10^{-6}	9.057×10^{-6}	4.6×10^{-9}	0.05%

According to [5], for large values of their arguments, it is possible to consider $I_0(x) \simeq \exp(x)/\sqrt{(2\pi x)}$ and $\operatorname{erfc}(x) \simeq \exp(-x^2)/\sqrt{(\pi)}x$. Hence, from (7) one obtains the following approximate expression (with increasing accuracy for larger SNR values):

$$\begin{aligned} \mathrm{BER}'_{approx} & \frac{1}{\sqrt{8\pi ab}} \frac{b+a}{b-a} \exp\left[-\frac{(b-a)^2}{2}\right] \\ &= \frac{\sqrt{2}+1}{\sqrt{8\pi\sqrt{2}}} \frac{1}{\sqrt{\gamma_b}} \exp\left[-(2-\sqrt{2})\gamma_b\right] \end{aligned} \tag{8}$$

The exact BER and the approximate expression (8) are compared, as functions of the SNR, in Fig. 2. As one can see, the simple approximation (8), which entails a very low computational complexity, is very accurate for BER values of interest.

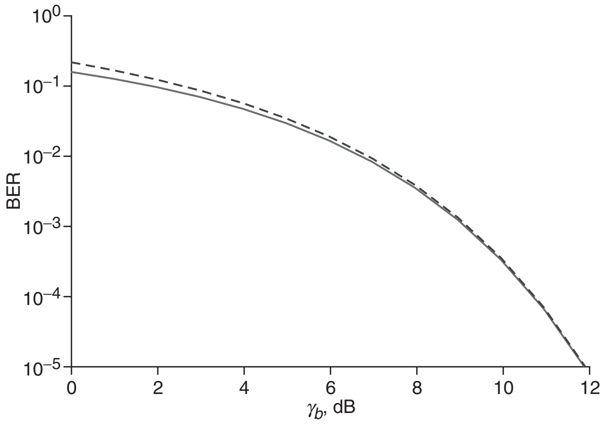

Fig. 2 *Comparison between approximate expression (8) and exact expression for BER of DQPSK transmission with Gray coding over AWGN channel*

——— exact

- - - - - approximation (8)

Conclusions: We have derived novel tight bounds for the BER of DQPSK with Gray coding based on recently proposed bounds for the Marcum Q-function. Owing to the symmetry of the obtained bounds, it was natural to use their arithmetic average as an approximate BER expression, from which a further simple approximate expression, characterised by a very low computational complexity, has been derived. It has been shown that the proposed approximate expressions are extremely accurate for all BER values of interest.

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