Transactions Letters

Does the Performance of LDPC Codes Depend on the Channel?

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***Abstract*—In this letter, we discuss the performance of low-density parity-check (LDPC) codes on memoryless channels. Using a recently proposed analysis technique based on extrinsic information transfer (EXIT) charts, we present an interpretation of the known fact that the bit-error rate (BER) performance of an ensemble of LDPC codes shows little dependence on the specific memoryless channel. This result has been partially observed in the literature for *symmetric* channels and is here extended to *asymmetric* channels. We conjecture and demonstrate that the performance of an ensemble of LDPC codes depends primarily and solely on the mutual information (MI) between the input and the output of the channel. As a validation of this conjecture, we compare the performance of a few LDPC codes with various rates for five representative memoryless (both symmetric and asymmetric) channels, obtaining results in excellent agreement with the EXIT chart-based prediction.**

***Index Terms*—Extrinsic information transfer (EXIT) charts, iterative decoding, low-density parity-check (LDPC) codes, memoryless channel.**

I. Introduction

In the last decade, the introduction of new coding techniques has allowed to achieve near-capacity transmission over the additive white Gaussian noise (AWGN) channel. In particular, the codes invented in [1], i.e., low-density parity-check (LDPC) codes, were recently rediscovered and analyzed in [2]. The assumptions considered in [1] for the derivation of an iterative decoding algorithm are valid for a *memoryless* channel [3]. *Regular* LDPC codes, introduced in [1], have been extended to *irregular* LDPC codes, which show improved performance [4]. In particular, an ensemble of LDPC codes can be defined in terms of *degree distributions*, which describe statistically the internal structure of the codes in terms of *variable* and *check* node degree distributions [5]. In [6] and [7], a powerful analysis technique based on extrinsic information transfer (EXIT) charts—first developed in [8] for turbo codes (TCs)—is applied to LDPC codes. In [9], practical approximations of EXIT charts are proposed and used for LDPC code design. Recently, tight upper and lower bounds for EXIT charts have been derived in [10]–[12]. These bounds allow to find transmission conditions, in terms of mutual information (MI) between the input and the output of the channel, for which it is possible to guarantee convergence regardless of the specific channel. Nevertheless, these bounds do not completely reflect the actual behavior of the decoding process of LDPC codes, which, as the number of iterations increases, seems to converge to that of the binary erasure channel (BEC) bound regardless of the specific channel [13]. This behavior, which has been experimentally observed, seems related to the fact that in the last iterations the bit error rate (BER) is low.

In this paper, we show that a performance analysis of LDPC codes based on simple EXIT charts suggests that the behavior of an *ensemble* of LDPC codes (i.e., a set of codes with given degree distributions) does not depend appreciably on the particular memoryless channel, but only on the MI between the input and the output of the channel. We then conjecture that a code from a given ensemble will exhibit *similar* convergence threshold and performance on any memoryless channel—in other words, different memoryless channels exhibit minor performance differences for a given value of MI. We support this conjecture by simulation results relative to various LDPC codes and several memoryless channels. The considered channels are both *symmetric*—binary-input AWGN channel, binary symmetric channel (BSC), and BEC—and *asymmetric*—binary asymmetric channel (BAC) and Z channel (ZC). This conjecture confirms the early remark in [5], where it was observed that LDPC codes optimized for the AWGN channel show good performance for other memoryless channels, such as BSC and BEC. In [14], the potential of iterative decodable codes to achieve a performance close to the capacity on memoryless channels is discussed, focusing on per-channel code optimization rather than on the universality of single codes. Our conjecture, moreover, generalizes the results in [15] and [16], where the authors show that the performance of LDPC codes over any Gaussian channel, not necessarily AWGN, depends only on the MI between the input and the output of the channel. We remark that the EXIT-chart-based analysis of LDPC codes assumes that the graphs of the corresponding LDPC codes do not contain short cycles. This condition can be achieved, for example, by choosing a sufficiently large codeword length.

II. EXIT Charts of LDPC Codes

In an EXIT-chart-based analysis of LDPC codes, the set of variable nodes and the set of check nodes are treated as blocks which process *soft* messages at their input and generate *soft* messages at their output [8], [9]. The ensemble of messages generated by a group of nodes, either variable or check, will be referred to as "message set." In Fig. 1, a pictorial representation of the receiver shows two different blocks: block **A** represents

Paper approved by S. G. Wilson, the Editor for Coding Theory and Applications of the IEEE Communications Society. Manuscript received December 10, 2004; revised August 15, 2005. This work was presented in part at the International Symposium on Information Theory and its Applications (ISITA), Parma, Italy, October 2004.

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Digital Object Identifier 10.1109/TCOMM.2006.885042

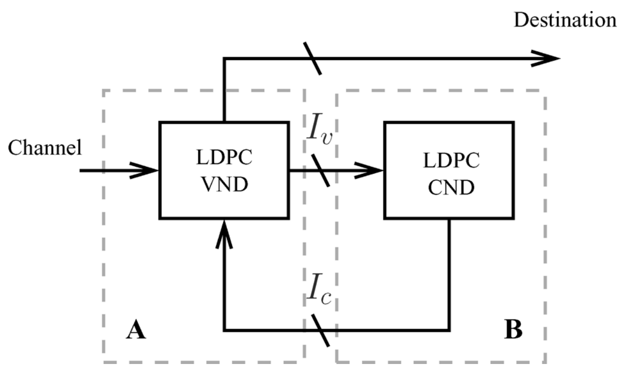

Fig. 1. Block diagram of an iterative receiver for LDPC-coded transmissions.

the set of all variable nodes and is denoted as variable node detector (VND); block **B** represents the set of all check nodes, and is denoted as check node detector (CND). Each message set, at the input or output of a block, can be viewed as the output of a channel with the actually transmitted codeword at its input. The message set can therefore be interpreted as a stochastic function of the transmitted codeword. For each message set, the MI between the transmitted codeword and the message set can then be computed.[1] In Fig. 1, I_v denotes the MI at the output of the VND and I_c the MI at the output of the CND. The underlying assumption is that the MI at the output of a block is a function of the MI at the input, regardless of the actual statistical distribution of the messages. In [8], sufficient conditions for an accurate analysis are outlined, and it is shown that a simple Gaussian approximation for the messages (seen as random variables) at the output of the variable nodes is effective.

The decoding process can be described as a *recursive* computation of the MI between the VND and the CND. Plotting the MI at the output of each block as a function of the MI at the input, i.e., drawing the corresponding EXIT chart, it is possible to predict the convergence characteristics of the considered system. In particular, it is possible to compute the so-called *convergence threshold*, defined as the value of the MI between the input and the output of the channel above which the recursive update of the MI between blocks **A** and **B** converges to one. If a message set at the output of a block has MI equal to one, then there exists a deterministic function of this message set that allows error-free computation of the transmitted bits. The convergence to one is possible if there is an "*open tunnel*" between I_v and I_c. In Fig. 2, the EXIT chart of a regular (3,6) LDPC code [1], [17] transmitted over an AWGN channel, with signal-to-noise ratio[2] (SNR) equal to 1.8 dB, is shown. The upper function I_v is the MI at the output of the VND as a function of the MI at its input; the lower function I_c^{-1} is the inverse of the MI at the output of the CND as a function of the MI at its input. The use of the inverse of the function I_c is expedient for a graphical study of the recursive evolution of the MI, depicted in Fig. 2 as a dashed line. It can be shown that the previous considerations imply that I_v depends on the VND distribution and on the channel, whereas I_c depends only on the CND distribution [5], [6]. Note that,

[1]Throughout this paper, the information bits are assumed to be equiprobable.

[2]The SNR is defined as the ratio of the information bit energy to the one-sided noise power spectral density.

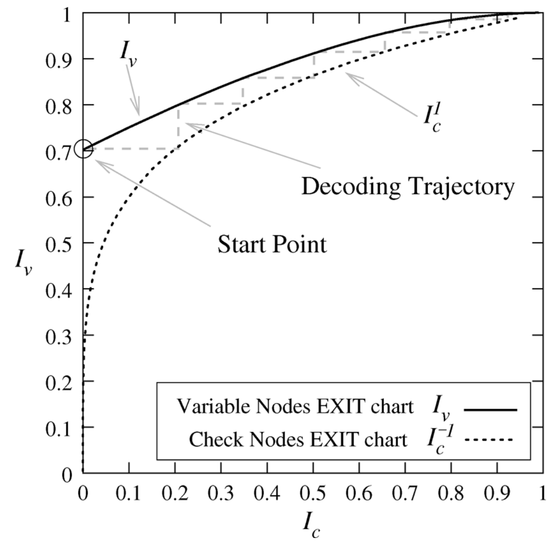

Fig. 2. EXIT chart for a regular (3,6) LDPC code transmitted over an AWGN channel with SNR equal to 1.8 dB. The dashed line represents the evolution trajectory of the MI.

since the check nodes at the very first iteration have no information to pass to the variable nodes, at the beginning of the iterative decoding process I_c is equal to 0. The start point of the decoding trajectory on the EXIT chart is therefore the point $(0, I_v(0))$, as shown in Fig. 2.

III. Start Point in EXIT Charts

What is the meaning of the start point of the decoding trajectory in an EXIT chart? The messages at the output of variable nodes at the very first iteration correspond to the logarithmic-likelihood ratios (LLRs), based on channel observations, of the transmitted symbols [1]. These quantities are *sufficient statistics* for an optimal decision on the transmitted sequence. This means that the MI between the transmitted binary sequence and these LLRs is equal to the MI between the transmitted binary sequence and the channel output, which, since the transmitted bits are assumed to be 0 or 1 with probability 1/2, is also known as the *constrained-input channel capacity* and is denoted here as $C_{\rm ci}$. Hence, at the first iteration, the MI generated at the output of the VND is $I_v = C_{\rm ci}$. As stated in Section II, this value corresponds to the point $(0, I_v(0))$, i.e., the start point of the EXIT-chart decoding trajectory.

Since simple EXIT-chart-based analyses assume that the MI at the output of a block is independent of the particular distribution of the messages, but it depends only on the MI at its input, the CND EXIT curve (i.e., I_c) does not depend on the particular channel. Nevertheless, the VND EXIT curve may depend on the channel. Experience suggests, however, that the VND EXIT curve (i.e., I_v) depends on the channel through $C_{\rm ci}$ only, whereas it depends weakly on the particular channel type. This is taken into account in the practical approximations currently used for LDPC code design in [9], where the author expresses I_v as a function of the MI between the input and the output of the channel and the MI of the messages coming from the CND. Since, as previously stated, $C_{\rm ci} = I_v(0)$, i.e., the start point of the EXIT-chart decoding trajectory, it follows that the entire function $I_v(I)$ can be characterized by this start point.

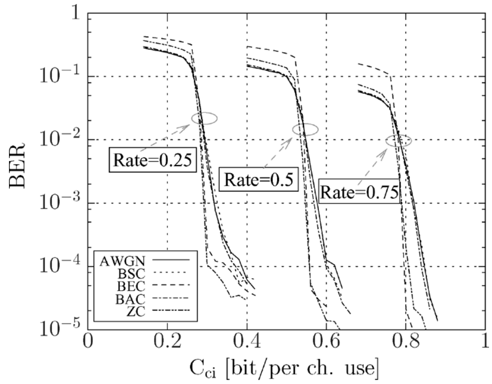

Fig. 3. BER versus constrained-input capacity for three LDPC codes, characterized by rates 1/4, 1/2, and 3/4, respectively. For each code, the performance for transmission over five memoryless channels is shown. The codeword length is 10 000 and the codes are optimized for transmission on the AWGN channel.

Assuming that the EXIT-chart-based analysis is accurate, one can conclude that the convergence of the decoding process for an ensemble of LDPC codes, described by their degree distributions, depends only on the constrained-input channel capacity and not on the particular channel—*provided that* the channel is memoryless. This means that *if* a randomly chosen code of a given ensemble shows, with high probability, good BER performance when transmitted over a memoryless channel with given $C_{\rm ci}$, *then* this code, with high probability, will guarantee good BER performance also when used for transmission over any other memoryless channel with equal $C_{\rm ci}$.

Note that these arguments are heuristic, and their accuracy is strictly related to the accuracy of the EXIT-chart-based analysis. In Section IV, simulation results will be presented to support our conjecture that LDPC codes belonging to the same ensemble and transmitted over a memoryless channel show similar performance *regardless* of the specific channel.

IV. Numerical Results

We consider Monte Carlo simulation-based performance analysis of three LDPC codes transmitted over five different memoryless channels. The considered codes have rates 1/4, 1/2, and 3/4 and are generated starting from the degree distributions, optimized for the binary-input AWGN channel, found in [18]. The codeword length is set to 10 000 binary symbols in all cases. The considered memoryless channels are three symmetric channels (binary input AWGN channel, BSC, and BEC) and two asymmetric channels (BAC and ZC). We evaluate the BER performance of each considered code over each channel as a function of the constrained-input capacity. For the BAC, the transition probability $P\{0 \rightarrow 1\}$ is different from the transition probability $P\{1 \rightarrow 0\}$. Two parameters are then necessary to describe this channel (and to compute $C_{\rm ci}$). We choose to specify the ratio $t \triangleq P\{0 \rightarrow 1\}/P\{1 \rightarrow 0\}$ as a given constant parameter. The ZC can be interpreted as a particular instance of the BAC with $t = 0$. It is then possible to express $C_{\rm ci}$ for every channel as a function of a single parameter, namely, the SNR γ for the AWGN channel, the transition probability for the BSC, the erasure probability for the BEC, and the average transition probability $(P\{1 \rightarrow 0\} + P\{0 \rightarrow 1\})/2$ for both BAC and ZC (all denoted by p in the following expressions). Summarizing, the constrained-input capacities of the considered channels have the following well-known expressions:

$$
\begin{aligned}
C_{\rm ci}^{\rm AWGN} &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{(y-2\gamma)^2}{8\gamma}} \log_2 \frac{2}{1+e^{-y}} {\rm d}y \\
C_{\rm ci}^{\rm BSC} &= 1 + p_{\rm BSC}\log(p_{\rm BSC}) + (1-p_{\rm BSC})\log(1-p_{\rm BSC}) \\
C_{\rm ci}^{\rm BEC} &= 1 - p_{\rm BEC} \\
C_{{\rm ci}|t}^{\rm BAC} &= \frac{1}{2(1+t)} \\
&\quad \times \Bigg\{ 2(1+t) + pt(1+t)\log\left[\frac{pt(1+t)}{(1+t)(1-p+pt)}\right] \\
&\quad + (1+t)(1-p)\log\left[\frac{(1+t)(1-p)}{(1+t)(1-p+pt)}\right] \\
&\quad + p(1+t)\log\left[\frac{p(1+t)}{(1+t)(1+p-pt)}\right] \\
&\quad + (1-pt)(1+t)\log\left[\frac{(1+t)(1-pt)}{(1+t)(1+p-pt)}\right] \Bigg\} \\
C_{\rm ci}^{\rm ZC} &= C_{{\rm ci}|t=0}^{\rm BAC} = \frac{1}{2}\left[2 + p\log p - (1+p)\log(1+p)\right].
\end{aligned}
$$

Incidentally, we remark that, for symmetric channels, $C_{\rm ci}$ equals the unconstrained capacity [3].

The decoding process stops if a codeword is obtained or if a maximum allowed number of 100 iterations is reached. In both cases, binary symbol decisions are made according to the final LLR value of each symbol, computed as the sum of all of the messages sent to its corresponding variable node at the last iteration. The receiver is assumed to know the channel statistics.

In Fig. 3, the BER curves of all three codes, transmitted over the considered five channels, are shown as a function of $C_{\rm ci}$—note that $C_{\rm ci}$ can assume values between 0 and 1, since the transmitted symbols are binary. For the BAC, the ratio $t = 3$ is chosen as a representative value. From the results in Fig. 3, one can conclude that the convergence threshold, in terms of $C_{\rm ci}$, basically depends only on the code, and, in a very limited way, on the channel. Interestingly, this conjecture holds for asymmetric channels as well. The slight differences between the BER curves relative to the same code are not predicted by the EXIT-chart-based analysis. In fact, the BER curves depend on the actual code structure, which may contain short cycles [5], and on the statistical distribution of the LLRs at the output of the channel, which are not taken into account by the EXIT charts. Moreover, by considering Fig. 3 one can quantify the actual difference between the performance of a code transmitted over the considered channels in terms of small fractions of bits per channel use (within a few hundredths).

V. Concluding Remarks

In this paper, we have conjectured and demonstrated that LDPC codes which are good for a particular memoryless channel are also good for any memoryless channel, in the sense that they guarantee similar BER performance in the same $C_{\rm ci}$ region, regardless of the channel type. Monte Carlo simulation has been presented to support this conjecture. Our considerations have been supported by results in terms of BER versus the MI between the input and the output of the channel. This enables a characterization of coding gain in terms of bits per channel use. Since our conjecture is motivated by an approximate analysis method, one can expect that a given LDPC code will exhibit the same performance within a small fraction of bits per channel use over different memoryless channels. This means that good LDPC codes for memoryless channels could be collected in code libraries and reused for several different applications, thus separating the tasks of: 1) designing LDPC codes and 2) fitting them to the considered concatenated scenario.

More rigorous claims regarding our conjecture would involve the derivation of new bounds on EXIT curves of LDPC codes, which extend the results in [10]–[12], taking into account how the distribution of the messages varies at each iteration. These bounds are the subject of current investigation.

The results presented in this paper may also impact the design of LDPC codes to be used in a bit interleaved coded modulation (BICM) scheme, which maps binary symbols onto high-order modulation formats [19]. At the receiver side, a soft demapper could generate reliability values for the mapped bits to be passed to the LDPC decoder, which would treat them as channel outputs. In this case, the decoding process does not depend on the particular mapper or channel but only on the MI at the output of the soft demapper. Assuming that iterations between demapper and decoder are not performed or not useful, LDPC codes designed for a simple memoryless channel (e.g., BSC) will be a good choice also if mapped over high-order modulations.

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