Public Key (asymmetric) Cryptography

Luca Veltri
(mail.to: luca.veltri@unipr.it)

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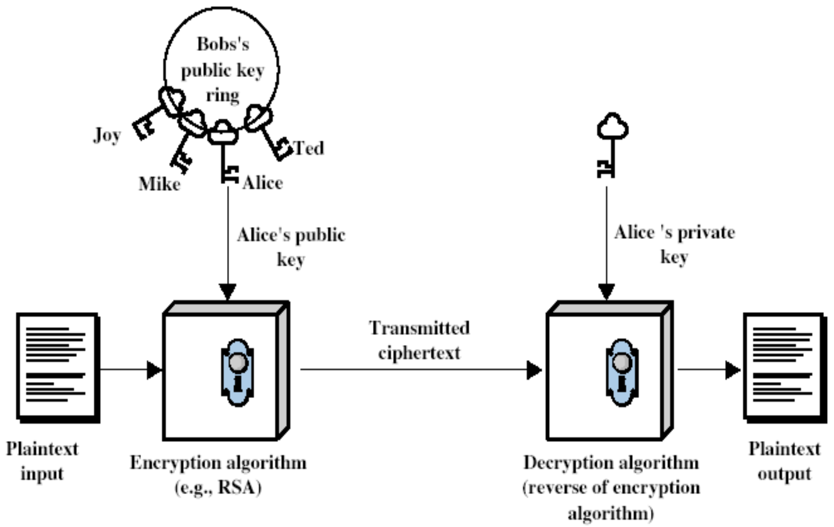
http://www.tlc.unipr.it/veltri

Public-Key Cryptography

- Also referred to as asymmetric cryptography or two-key cryptography
- Probably most significant advance in the 3000 year history of cryptography
  - **Public invention due to Whitfield Diffie & Martin Hellman in 1975**
    - at least that's the first published record
    - known earlier in classified community (e.g. NSA?)
- Is asymmetric because
  - **Who encrypts messages or verify signatures cannot decrypt messages or create signatures**
- Uses clever application of number theoretic concepts and mathematic functions rather than permutations and substitutions
- Complements rather than replaces secret key crypto

Public-Key Cryptography



Public-Key vs. Secret Cryptography

- All secret key algorithms do the same thing
  - **they take a block and encrypt it in a reversible way**
- All hash algorithms do the same thing
  - **they take a message and perform an irreversible transformation**
- Instead, public key algorithms look very different
  - **in how they perform their function**
  - **in what functions they perform**
- They all have in common: a private and a public quantities associated with a principal
- Example of public key algorithms:
  - **RSA, which does encryption and digital signature**
  - **El Gamal and DSS, which do digital signature but not encryption**
  - **Diffie-Hellman, which allows establishment of a shared secret**
  - **zero knowledge proof systems, which only do authentication**

Public-Key vs. Secret Cryptography (cont.)

- Public key cryptography can do anything secret key cryptography can do, but..
- The known public-key cryptographic algorithms are orders of magnitude slower than the best known secret key cryptographic algorithms
  - **are usually used only for things secret key cryptography can't do (or can't do in a suitable way)**
- Often it is mixed with secret key technology
  - **e.g. public key cryptography might be used in the beginning of communication for authentication and to establish a temporary shared secret key used to encrypt the conversation**

Public-Key vs. Secret Cryptography (cont.)

- With symmetric/secret-key cryptography
  - **you need a secure method of telling your partner the key**
  - **you need a separate key for everyone you might communicate with**
- Instead, with public-key cryptography, keys are not shared
- Public-key cryptography often uses two keys:
  - **a public-key, which may be known by anybody, and can be used to encrypt messages, or verify signatures**
  - **a private-key, known only to the recipient, used to decrypt messages, or sign (create) signatures**
  - **it is computationally easy to en/decrypt messages when key is known**
  - **it is computationally infeasible to find decryption key knowing only encryption key (and vice-versa)**
- Some asymmetric algorithms don't use keys at all!

Why Public-Key Cryptography?

- Can be used to:
  - **key distribution – secure communications without having to trust a KDC with your key (key exchange)**
  - **digital signatures –verify a message is come intact from the claimed sender (authentication)**
  - **encryption/decryption - secrecy of the communication (confidentiality)**
- Some algorithms are suitable for all uses, others are specific to one
- Note that public-key cryptography simplifies but not eliminates the problem of trusted systems and key management

Security of Public Key Schemes

- Security of public-key algorithms still relies on key size (as for secret-key algorithms)
- Like private key schemes brute force exhaustive search attack is always theoretically possible
  - **But keys used are much larger (>512bits)**
- A crucial feature is that the private key is difficult to determine from the public key
  - **security relies on a large enough difference in difficulty between easy (en/decrypt) and hard (cryptanalyse) problems**
  - **often the hard problem is known, its just made too hard to do in practise**
    - requires the use of very large numbers
    - hence is slow compared to private key schemes

Rivest, Shamir, and Adleman (RSA)

Rivest, Shamir, and Adleman

- by Rivest, Shamir & Adleman of MIT in 1977
- best known & widely used public-key scheme
- Based on exponentiation in a finite (Galois) field over integers modulo n
  - **nb. exponentiation takes $O((\log n)^3)$ operations (easy)**
- uses large integers (eg. 1024 bits)
- security due to cost of factoring large numbers
  - **nb. factorization takes $O(e^{\log n \log \log n})$ operations (hard)**
- The key length is variable
  - **long keys for enhanced security, or a short keys for efficiency**
- The plaintext block size (the chunk to be encrypted) is also variable
  - **The plaintext block size must be smaller than the key length**
  - **The ciphertext block will be the length of the key**
- RSA is much slower to compute than popular secret key algorithms like DES, IDEA, and AES

Some Arithmetic

- Relatively prime - means that two values do not share any common factors other than 1
  - **eg. 8 & 15 are relatively prime since factors of 8 are 1,2,4,8 and of 15 are 1,3,5,15 and 1 is the only common factor**
- totient function - $\phi(n)$ - tells how many numbers less than n are relatively prime to n; a.k.a. Euler's totient function
  - **to compute $\phi(n)$ need to count number of elements to be excluded**
    - e.g. $\phi(8) = |1,3,5,7| = 4$
  - **in general need prime factorization, but**
  - **Theo: If n is prime, the all integers less than n (that is: 1, 2, ..., n-1) are relatively prime to n; therefore $\phi(p) = p - 1$**
    - e.g. $\phi(37) = 36$
  - **Theo: If n is the product of two primes (p and q) then there are (p-1)(q-1) numbers relatively prime to that quantity, that is $\phi(n) = \phi(pq) = (p-1)(q-1)$**
    - e.g. f(21) = (3-1) x (7-1) = 2x6 = 12

Some Arithmetic

- a mod $n=r_a$ | $a=q_a \cdot n + r_a$
- a=b mod n means that (a mod n) = (b mod n)
  - **i.e. $a-b=k \cdot n$**
- properties:
  - **a op b mod n ≡ (a mod n) op (b mod n) = (a op b) mod n**
  - **with op = + , - , ***
- Some definitions in arithmetic modulo n
  - **complete set of residues is: 0..n-1**
  - **reduced set of residues is those numbers (residues) which are relatively prime to n**
    - **eg for n=10, complete set of residues is {0,1,2,3,4,5,6,7,8,9}**
    - **reduced set of residues is {1,3,7,9}**
  - **number of elements in reduced set of residues is the Euler Totient function $\phi(n)$**

Some Arithmetic

- Multiplicative inverse - the multiplicative inverse of a number x is the number we multiply x by to get 1
  - **with real numbers this is just 1/x**
  - **The multiplicative inverse of m mod n is u | u*m = 1 mod n**
    - u*m differs from 1 by a multiple of n, or u*m + v*n = 1
    - Euclid's algorithm can be used to solve this knotty problem. It only works if m and n are relatively prime

Multiplicative inverse - Example

- Find the inverse of 797 mod 1047?
- That is, find u such that u * 797 = 1 mod 1047, or u*797 + v*1047 = 1
- Using Euclid's algorithm
  - **we get u = -490 and v = 373**
  - **That is: -490 * 797 + 373 * 1047 = 1**
  - **So (u*m) is -490 * 797 = -390530**
  - **The multiplicative inverse (u) is -490**
  - **-390530 mod 1047 = 1 mod 1047**

RSA Algorithm

- First, you need to generate a public key and a corresponding private key:
  - **choose two large primes p and q (around 512 bits each or more)**
    - p and q will remain secret
  - **multiply them together (result is 1024 bits), and call the result n**
    - it's practically impossible to factor numbers that large for obtaining p and q
  - **choose a number *e* that is relatively prime (that is, it does not share any common factors other than 1) to ϕ(n)**
    - since you know p and q, you know $\phi(n) = (p-1)(q-1)$
  - **your public key is KU =<e,n>**
  - **find the number *d* that is the multiplicative inverse of e mod ϕ(n)**
  - **your private key is KR=<d,n> or KR=<d,p,q>**
- To encrypt a message m (< n), someone can use your public key
  - **$c = m^e \bmod n$**
- Only you will be able to decrypt c, using your private key
  - **$m = c^d \bmod n$**

Fermat's and Euler Theorems

- Fermat Theorem
  - **$a^{p-1} \bmod p = 1$**
    - where p is prime, and
    - a is not divisible by p, i.e. the gcd(a,p)=1
- Euler Theorem
  - **a generalisation of Fermat's Theorem**
  - **$a^{\varnothing(n)} \bmod n = 1$**
    - where gcd(a,n)=1
  - **eg.**
    - a=3;n=10; ø(10)=4;
    - hence $3^4 = 81 = 1 \bmod 10$
    - a=2;n=11; ø(11)=10;
    - hence $2^{10} = 1024 = 1 \bmod 11$
- Corollary from Eulers's Theorem
  - **$a^{k\varnothing(n)+1} \bmod n = a$**

Why RSA Works

- Because of Euler's Theorem:
  - $a^{k\varnothing(n)+1} \bmod n = a$
    - where gcd(a,n)=1
- In RSA have:
  - **n=p.q**
  - **ø(n)=(p-1)(q-1)**
  - **carefully chosen e & d to be inverses mod ø(n)**
  - **hence e·d=1+k·ϕ(n) for some k**
- Hence:
  $c^d = (m^e)^d = m^{1+k\varnothing(n)} = m \bmod n$

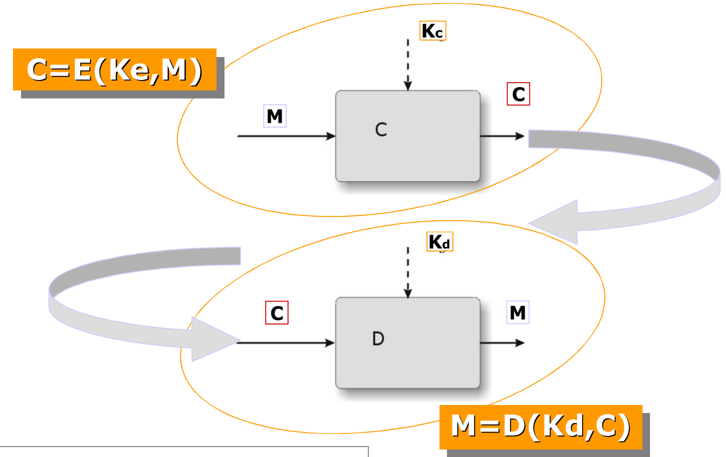
RSA Key Setup

- Each user generates a public/private key pair by:
  - **selecting two large primes at random** `p,q`
  - **computing their system modulus** `n = p q`
    - note `ø(n)=(p-1)(q-1)`
  - **selecting at random the encryption key** `e`
    - where `1<e<ø(n), gcd(e,ø(n))=1`
  - **solve following equation to find decryption key** `d`
    - `e d = 1 mod ø(n) and 0≤d≤n`
- Publish their public encryption key: KU={e,n}
- Keep secret private decryption key: KR={d,p,q}

RSA Use

- To encrypt a message M the sender:
  - **obtains public key of recipient KU=<e,n>**
  - **computes: c=me mod n, where 0≤m<n**
- To decrypt the ciphertext c the owner:
  - **uses their private key KR=<d,n>**
  - **computes: m=c^d mod n**
- Note that the message m must be smaller than the modulus n (block if needed)

RSA



- **M** Blocco di testo in chiaro
- **C** Blocco di testo cifrato
- **Ke** Chiave cifratura (e.g. chiave pubblica Ku)
- **Kd** Chiave decifratura (e.g. chiave privata Kr)

RSA Example

RSA setup

- select primes: p=17 & q=11
- compute n = pq =17×11=187
- compute ø(n)=(p–1)(q-1)=16×10=160
- select e : gcd(e,160)=1; choose e=7
- determine d: de=1 mod 160 and d < 160 Value is d=23 since 23×7=161= 10×160+1
- publish public key KU={7,187}
- keep secret private key KR={23,187}={23,17,11}

RSA Example (cont)

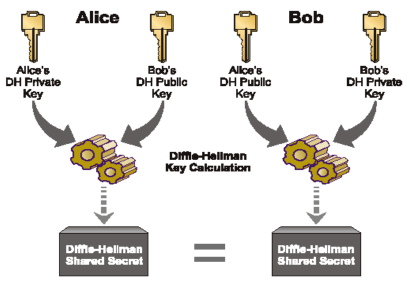
RSA encryption/decryption:

- given message `M = 88` (nb. `88<187`)
- encryption:
  C = 88^7 mod 187 = 11
- decryption:
  M = 11^{23} mod 187 = 88

RSA Security

- three approaches to attacking RSA:
  - **brute force key search (infeasible given size of numbers)**
  - **mathematical attacks (based on difficulty of computing ø(N), by factoring modulus N)**
  - **timing attacks (on running of decryption)**

Diffie-Hellman



Diffie-Hellman

- First public-key type scheme proposed
- By Diffie & Hellman in 1976 along with the exposition of public key concepts
  - **now know that James Ellis (UK CESG) secretly proposed the concept in 1970**
    - predates RSA
  - **less general than RSA: it does neither encryption nor signature**
- Is a practical method for public exchange of a secret key
  - **allows two individuals to agree on a shared secret (key)**
  - **It is actually used for key establishment**
- Used in a number of commercial products

Diffie-Hellman Setup

Diffie-Hellman setup:

- all users agree on global parameters:
  - **p = a large prime integer or polynomial**
  - **g = a primitive root mod p**
- each user (eg. A) generates their key
  - **chooses a secret key (number):** $x_A < p$
  - **compute their** public key**:** $y_A = g^{x_A} \bmod p$
- each user makes public that key y_A

Diffie-Hellman Key Exchange

Key exchange:

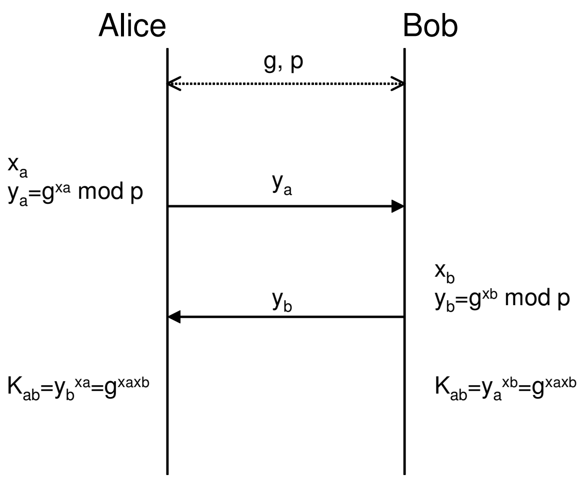
- Shared key K_{AB} for users A & B can be computed as:

  $K_{AB} = g^{x_A \cdot x_B} \bmod p$

  $= y_A^{x_B} \bmod p$ (which B can compute)

  $= y_B^{x_A} \bmod p$ (which A can compute)
- K_{AB} can be used as session key in secret-key encryption scheme between A and B
- Attacker must solve discrete log

Diffie-Hellman



Diffie-Hellman - Example

- users Alice & Bob who wish to swap keys:
- agree on prime `p=353` and `g=3`
- select random secret keys:
  - **A chooses x_A=97, B chooses x_B=233**
- compute public keys:
  - $y_A = 3^{97} \bmod 353 = 40$ **(Alice)**
  - $y_B = 3^{233} \bmod 353 = 248$ **(Bob)**
- compute shared session key as:

$K_{AB} = y_B^{x_A} \bmod 353 = 248^{97} = 160$ **(Alice)**
$K_{AB} = y_A^{x_B} \bmod 353 = 40^{233} = 160$ **(Bob)**

Zero Knowledge Proof Systems

- Only do authentication
  - **prove that you know a secret without revealing the secret**
- RSA is a zero knowledge system
- There are zero knowledge systems with much higher performance
- Example (Isomorphic graphs):
  - **Alice defines two large (say 500 vertices) isomorphic graphs G_A, G_B**
  - **G_A and G_B become public, but only Alice knows the mapping**
  - **to prove her identity to Bob, Alice find a set of isomorphic graphs $G_1, G_2, .., G_k$**
  - **Bob divides the set into two subset T_A and T_B**
  - **Alice shows to Bob the mapping between each $G_i \in T_A$ and G_A, and between each $G_j \in T_B$ and G_B**

Security uses of public key cryptography

- Transmitting over an insecure channel
  - **each party has a <public key, private key> pair (Ku,Kr)**
  - **each party encrypts with the public key of the other party**

encrypt m_A using Ku_B ⟶ **decrypt m_A using Kr_B**
decrypt m_B using Kr_A ⟵ **encrypt m_B using Ku_A**

- Secure storage on insecure media
  - **encrypt with public key, decrypt with private key**
  - **useful when you can let third party to encrypt data**
- Peer Authentication
  - **public key gives the real benefit**
  - **no n(n-1)/2 keys are needed**

encrypt r using Ku_B ⟶ **decrypt to r using Kr_B**
⟵ **r**

Security uses of public key cryptography

- Data authentication (Digital signature)
  - **based on cryptographic checksum**
- Key establishment
  - **e.g. Diffie-Hellman**
- Note
  - **Public key cryptography has specific algorithm for specific function such as**
    - data encryption
    - MAC/digital signature
    - peer authentication
    - key establishment

Vantaggio dei sistemi a chiave pubblica

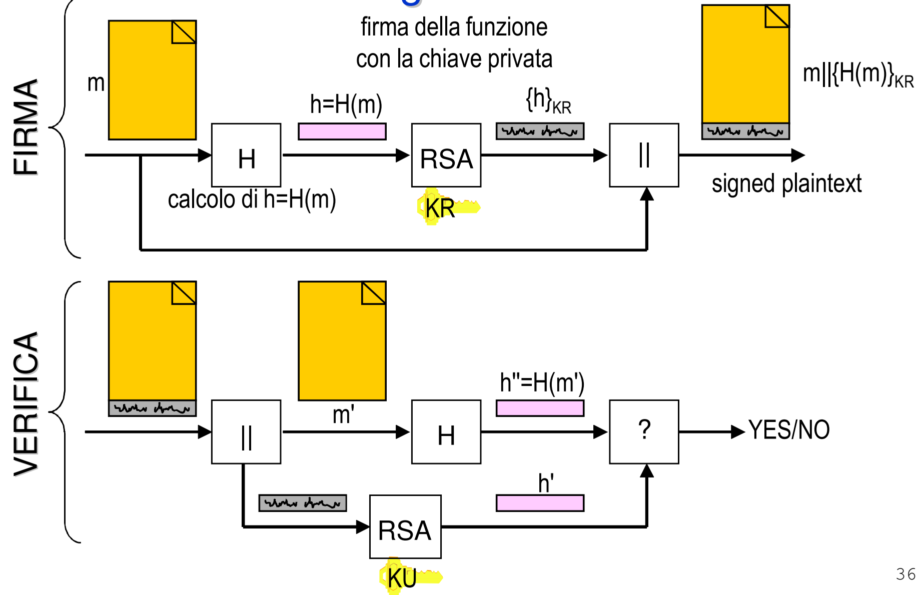
- Ogni utente deve mantenere solo un segreto (la propria chiave privata)
- Le chiavi pubbliche degli altri utenti possono essere mantenuti tramite infrastrutture intermediarie sicure (PKI)
- Il numero delle chiavi è proporzionale a N per la comunicazione reciproca tra N utenti

Digital signature and digital certification

Digital Signature

- Digital Signature is an application in which a signer, say "Alice," "signs" a message m in such a way that
  - **anyone can "verify" that the message was signed by no one other than Alice, and**
  - **consequently that the message has not been modified since she signed it**
- i.e. the message is a true and correct copy of the original
- The difference between digital signatures and conventional ones is that digital signatures can be mathematically verified
- The typical implementation of digital signature involves a message-digest algorithm and a public-key algorithm for encrypting the message digest (i.e., a message-digest encryption algorithm)

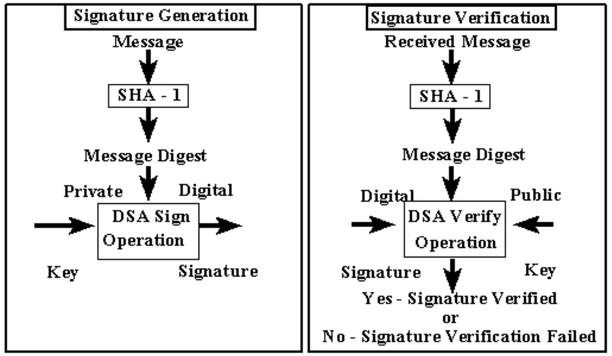
Firma Digitale con RSA



Digital Signature Standard (DSS)

- DSS (Digital Signature Standard)
- Proposed by NIST (U.S. National Institute of Standards and Technology) & NSA in 1991
  - **FIPS 186**
- Based on an algorithm known as DSA (Digital Signature Algorithm)
  - **is a variant of the ElGamal scheme**
  - **uses 160-bit exponents**
  - **creates a 320 bit signature (160+160) but with 512-1024 bit security**
  - **uses SHA/SHS hash algorithm**
- Security depends on difficulty of computing discrete logarithms

DSS Operations



DSA Key Generation

- have shared global public key values (p,q,g)
  - **L is the key length**
    - L = 1024 or more, and is a multiple of 64
  - **a large prime p**
  - **choose q, a 160 bit prime factor of p-1**
    - actually long as the hash H
  - **choose g | $g=h^{(p-1)/q}$**
    - where $h<p-1$, $h^{(p-1)/q} \bmod p > 1$
    - for some arbitrary h with 1<h<p-1
- choose $x<q$
- compute $y = g^x \bmod p$
- public key = (p,q,g,y)
- private key = x

DSA Signature Creation

- to sign a message M the sender generates:
  - **a random signature key k, k<q**
    - N.B.: k must be random, be destroyed after use, and never be reused
- computes the message digest:

  h = SHA(M)
- then computes signature pair:

  $r = (g^k \bmod p) \bmod q$

  $s = k^{-1}(h+x\cdot r) \bmod q$
- sends signature (r,s) with message M

DSA Signature Verification

- having received M & signature (r, s)
- to verify a signature, recipient computes:

  $w = s^{-1} \bmod q$

  $v = (g^{hw \bmod q}\, y^{rw \bmod q} \bmod p) \bmod q$
- if v=r then signature is verified

- proof

  $v = (g^{hw \bmod q}\, y^{rw \bmod q} \bmod p) \bmod q =$

  $= (g^{w(h+xr) \bmod q} \bmod p) \bmod q =$

  $= (g^{k} \bmod p) \bmod q =$

  $= r$

Digital Certification

- Digital certification is an application in which a certification authority "signs" a special message m containing
  - **the name of some user, say "Alice," and**
  - **her public key**

  in such a way that anyone can "verify" that the message was signed by no one other than the certification authority and thereby develop trust in Alice's public key
- The typical implementation of digital certification involves a signature algorithm for signing the special message